A STUDY OF COMMON FIXED POINT THEOREMS IN METRIC **AND** FUZZY METRIC SPACES

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

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A\ IN

„ **MATHEMATICS**

By

JA ***VID ALT***

UNDER THE SUPERVISION OF

Prof. M. IMDAD

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

December, 2007

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(Declaration By tfie Candidate

I declare that the thesis entitled ***"A study of common fixed point theorems in metric andfuzTy metric spaces"*** is the compilation of my own work carried out under the supervision of **Dr. Mohd. Imdad,** Professor in the Department of Mathematics, A. M. U., Aligarh.

To the best of my knowledge and belief I further declare that this thesis does not contain any part of the work which had been submitted for the award of any degree either in this university or in other institution without proper citation.

Dated: 17*' Dec, 2007 **(Javid All)**

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This is to certify that the material presented in the thesis entitled "A ***Study of common fixed point theorems in metric and fuzzy metric spaces***" is the research work of ***Mr. Javid Ali*** carried out under my supervision. This work is more than adequate for the partial fulfillment for the award of the degree of Doctor of Philosophy in Mathematics.

I fiirther certify that the work has not been submitted either partly or fully to any other university or institution for the award of any degree.

Dr. M. Imdad
(Supervisor)

A CKNO WLED GHENT

First and foremost, I must bow in reverence to Almighty 'ALLAH' the cherishers and sustainer to whom we entirely owe our ail capabilities, strength, courage and knowledge without his blessings nothing could be done.

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Dated: 17th Dec., 2007 ^ ' ' ^ (Javid Ali)

CONTENTS

Preface

Fixed point theory is a rich, interesting and highly apphed branch of nonlinear functional analysis which has always greatly facilitated in the existence theory of differential equations, integral equations, functional equations, partial differential equations and other related areas. Fixed point theory also finds fruitful applications in eigenvalue problems, boundary value problems, approximation theory, game theory, optimal control, variational inequalities, equilibrium problems and complementarity problems. Fixed point theorems are also used in new areas of mathematical applications other than mathematics which includes mathematical economics, fluid flow, random diflFerential equations, image compression, fractals etc.

The first ever fixed point theorem for contraction mappings in metric spaces was coined by S. Banach which is now popularly named as Banach contraction principle. By now, this principle has become one of the most fundamental and powerful tools of nonlinear analysis due to its wide range of apphcations to nonlinear equations arising in physical and biological processes ensuring the existence and uniqueness of solutions. Though Banach established his celebrated contraction principle in 1922, but its all extensions and generalizations were proved after 1968 when Kannan [84] established some fixed point theorems in metric spaces for discontinuous self mapping. The work of Kannan [84] has inspired extensions and generalizations of contraction principle which is still flourishing in several directions.

In 1976, Jungck [70] proved an extension of Banach contraction principle employing a pair of self commuting mappings which has greatly influenced the researchers thereafter. By now there exists vast literature around Jungck theorem which includes its extension to multi-valued mappings and also to hybrid mappings. Another metrical fixed point theorem relevant to the present thesis is a fixed point theorem due to Assad and Kirk [5] proved for nonself mappings whose domains are closed subsets of metrically convex metric spaces. In recent years the fixed point theorem due to Assad and Kirk [5] has been generalized and improved in various ways by several authors. To mention a few, one can cite [6,7,8,54-56,88,91].

The purpose of our work is four fold:

(a) To prove results on coincidence and common fixed points for pairs of single-valued and hybrid mappings under strict contraction in semi-metric (potent semi-metric) spaces (Chapters 2 and 3).

(6) To prove coincidence and common fixed point theorems for pairs of single-valued and hybrid mappings satisfying the property *{E.A)* and common property *{E.A)* in metric spaces (Chapters 4 and 5).

(c) To prove common fixed point theorems for nonself mappings by altering distances between the points in metrically convex metric spaces (Chapter 6).

(d) To prove results on coincidence and common fixed point theorems for self mappings in fuzzy metric spaces iising a newly introduced implicit function (Chapter 7).

The present thesis consisted of seven chapters. Each chapter is divided into various sections. The numbers hke (3.2) indicates Equation 2 of Chapter 3 whereas 5.4.1 indicates Theorem (or Lemma/Proposition/Remark/Corollary/Definition) 1 of Section 4 of Chapter 5. The hypotheses, conditions or conclusions of Chapters 1 to 7 are denoted by a„, 6„, c„, ...y„ respectively where *n* varies. As usual the numbers in brackets refer to the references listed in the bibliography. The first section of each chapter provides an introduction to its contents.

As usual, Chapter 1 is introductory in nature wherein we have discussed the historical development of fixed point theory and visited relevant preliminary concepts, definitions and important results to our subsequent discussions.

The chapter second is devoted to certain results on coincidence and common fixed points of single-valued mappings satisfying the property *{E.A)* under strict contractions in semi-metric (potent semi-metric) spaces. Section 2.2 mainly presents results on coincidence and common fixed points for a pair of weakly compatible self mappings. This section concludes with an illustrative example substantiating the utility of our preceeding results. In Section 2.3, we further prove coincidence and common fixed point theorems for two pairs of mappings employing a suitable control function besides furnishing an illustrative example to our earUer results.

In Chapter 3, we prove some common fixed point theorems for two pairs of hybrid mappings satisfying the common property *(E.A)* in semi-metric (potent semi-metric) spaces wherein the contraction condition is again of strict type. In process the hybrid fixed point theorems due to Kamran [82], Singh and Hashim [143], Liu et al. [94], Kasahara and Rhoades [86] and several others are in turn extended to semi-metric spaces while the results of Chapter 2 (of this thesis) are also generalized for pairs of hybrid mappings. We also discuss distinction between the terms symmetric, semi-metric and potent semi-metric.

Chapter 4 is mainly devoted to two general coincidence and common fixed point theorems for pairs of self mappings satisfying the property *(E.A)* and the common property *{E.A)*. In Section 4.2, we describe implicit function due to Popa [122] and use the same to prove a general fixed point theorem without any requirement of the containment on the ranges of the involved mappings. Section 4.4 deals with some examples in support of results proved in earlier section. A new class of implicit functions is introduced in Section 4.5 which covers almost all well known contraction conditions besides admitting new ones. Finally, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying the common property *(E.A)*. Our results proved in Section 4.6 generalize and improve almost all well known theorems especially from [2,21,23-25,34,36,46,50,53,57-59,61,69,72,83,89, 94,99,105,138,145].

Chapter 5 deals with common fixed point theorems for pairs of single-valued mappings along with a general result on a sequence of hybrid mappings via implicit functions. In Section 5.2, we define an implicit function which covers a host of well known contraction conditions and different from previous two imphcit functions considered in Chapter 4. We also furnish several examples enjoying the format of new implicit function. Section 5.3 is devoted to results on common fixed point under strict contractions whereas in Section 5.4 some examples are furnished to illustrate the results proved in earlier section. In concluding section, we prove a general fixed point theorem for a sequence of hybrid mappings employing imphcit function due to Popa [125] which generaUzes several previously known relevant results.

In Chapter 6, we prove results on coincidence and common fixed points of nonself mappings in metrically convex metric spaces. In Section 6.2, we introduce the notion of reciprocal continuity in nonself setting and prove a related result. Section 6.3 deals with some results on coincidence and common fixed points for reciprocally continuous nonself mappings which generalize earlier results due to Assad [7], Imdad et al. [56], Khan and Bharadwaj [88], Khan et al. [91] and many others. In the last section, we define Cg-commutativity for nonself mappings and compare it with other related conditions of weak commutativity. As an application of our main result in this chapter, we prove a common fixed point theorem in Banach spaces.

In the final chapter, we prove some common fixed point theorems for self mappings in fuzzy metric spaces in sense of Kramosil and Michalek [92] via a newly introduced implicit function. In Section 7.2, we collect the related definitions, results and examples. We also discuss the independence of certain types of weak conditions of commutativity. As indicated earlier, in Section 7.3, we define a new class of implicit functions and furnish several illustrative examples. In Section 7.4, a general coincidence and common fixed point theorem for two pairs of weakly compatible mappings satisfying a newly introduced implicit function is proved besides deriving some others related results along with some illustrative examples. As an application of main theorem of this chapter, we prove a fixed point theorem for four finite families of self mappings. In process om main result generalizes earlier results due to Imdad and Ali [52], Vasuki [152], Chugh and Kumar [22], Singh and Jain [140] and some others. We also observe that some of our corollaries are new to the existing literature of the fuzzy metric fixed point theory.

In the end, a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred to in this thesis.

LIST OF RESEARCH PAPERS

1. Coincidence and fixed points in symmetric spaces under strict contractions, J. Math. Anal. Appl. 320 (2006), 352-360.

2. A general fixed point theorem for hybrid contractions via implicit functions, Southeast Asian Bull. Math. 31 (2007), 73-80.

3. Jungck's common fixed point theorem and E.A property. Acta Math. Sinica, English Series 24 (2008), published online.

4. A general fixed point theorem in fuzzy metric spaces via an implicit function, J. Appl. Math. Informatics, accepted.

5. An implicit relation implies several contraction conditions, submitted after revision.

6. Reciprocal continuity and common fixed points of nonself mappings, submitted after revision.

7. Common fixed points of hybrid mappings in semi-metric spaces under strict contractions, communicated.

8. Common fixed point theorems revisited via common property E.A, communicated.

Chapter 1

PRELIMINARIES

1.1. INTRODUCTION

The great French mathematician H. Poincare (1854-1912) was, in fact, one of the founders of fixed point approach who had deep insight into future apphcations of the approach to diverse fields including mathematical analysis and classical mechanics and this led him to take an active role into the development of fixed point approach. The next pioneer contributor is the Dutch mathematician L.E.J. Brouwer (1881-1966) who initiated fixed point approach by proving a concrete result now popularly known as Brouwer fixed point theorem. He also introduced deep topological notions like 'homotopy' and 'degree of a mapping' and utilize them in fixed point context. He also proved the fixed point theorems for a square and for a sphere whose extension to n-dimensional Euclidean space is indeed the Brouwer's fixed point theorem. The Brouwer's fixed point theorem is not a result but a subject in itself. Since its many extensions are powerful tools in establishing the existence of solutions of several problems, various researchers have been studying its further extentions and generalizations whose details will be discussed in Section 1.2. On this theme, the survey article entitled 'Eighty years of the Brouwer's fixed point theorem' of Park [116] also deserves special mention.

Fixed point theory has always played a central role in the problems of functional analysis and topology has been involved deeply in both the study of fixed point theory and more directly to problems in analysis in a wide variety of ways, in fact, Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. It is a sub-branch of the functional analytic theory in which geometric conditions on the mappings and/or underlying spaces play a crucial role. Although it has a purely metric facet. It is also a major branch of nonlinear functional analysis with close ties to Banach space geometry.

By a fixed point of a self mapping T defined on a nonempty set X, one means to know whether one or more points of X are invariant under T, i.e. $Tx = x$ for some X e X. However, in case of nonself mapping T from X to F the same definition works provided X r\Y ^ 0. Fixed points of a family of mappings T with domain X and range Y with F D X 7^ 0 are those elements of X that are invariant by each element of J^. In a wide sense, by a fixed point theorem one means a statement which asserts that imder what conditions a mapping of X into Y admits one or more points of X for which $Tx = x$.

Since initial fixed point theorem of Brouwer [14], a host of many classical fixed point theorems were proved which inspire rigorous research activities around them. Most of the theorems are well known to specialists in fixed point theory. Due to the limitation of the spgice it is not possible to dehneate them all. To mention a few, we enlist the following:

(ai) Brouwer fixed point theorem ([14, 1912]),

(02) Banach contraction principle ([9, 1922]),

(03) Schauder fixed point theorem ([134, 1930]),

(^4) Tychonoff fixed point theorem ([151, 1935]),

(05) Sadovoski fixed point theorem ([129, 1967]),

(og) Ky Fan best approximation theorem ([31, 1969]),

(a-j) Nadler fixed point theorem ([101, 1969]),

(ag) Caristi fixed point theorem ([18, 1976]).

DEFINITION 1.1.1. A topological space X is said to have the fixed point property (in short FPP) if every continuous self mapping of X admits at least one fixed point.

Recall that any property is said to be topological if it is preserved under homeomorphism i.e., in case, a topological space X is homeomorphic to a topological space Y and if X has certain topological property then so does Y. Fortunately, fixed point property is also a topological property. To substantiate this, let us assume that

T '. X —>*Y he* any homeomorphism and let 5 : K —> F be any continuous mapping. Consider the mapping *F* = *T'^ST* from the set *X* into itself. Then *F* is continuous being the composition of continuous mappings *T~^,S* and *T*. If *XQ* remains fixed under *F* and *yo* denotes the image of *XQ* imder *T*, then

$$Fxo = T\text{-}\hat{}STXQ = XQ.$$

Operating *T* on both the sides, one gets

$$TT\text{-}\backslash STXQ) = Txo \Rightarrow STXQ = Txo$$

which shows that *TXQ* = yo is a fixed point of *S*.

In order to utilize fixed point property, in every case one is required to know the fixed point property of one member of an entire class of homeomorphic sets. Specifically, since this property is true for a square then it will remain true for a disc or any convex plane polygon.

It is interesting to note that fixed point property is closely related to the notion of retract. Recall that a subset y of a set *X* is said to be a retract of *X* if there exist a continuous mapping *T : X -^Y* preserving each point of the set *Y*. This mapping *T* is termed as retraction of the set *X* onto set *Y*. Note that a closed interval is a retract of a square whereas a circle is a retract of an annular region bounded by a pair of concentric circles.

The following theorem establishes the importance of this notion.

THEOREM 1.1.1. Let y be a retract of the set *X* and if *X* has the fixed point property then so does *Y*.

Since square has fixed point property and a closed interval is a retract of a square, the above theorem shows that closed interval has fixed point property. The same argument appUes to all other retracts of a square namely: circle, triangle and others.

The investigations in this thesis heavily bank on Banach contreiction principle, Nadler fixed point theorem and Jungck [70] common fixed point theorem, therefore these theorems and other related results will be discussed in detail latter.

It is never possible to give a complete description of a wide subject like fixed point theory in one or two paragraphs. However, for a comprehensive study of fixed point theory and its related results the books by Goebel [41], Goebel and Kirk [42], Granas and Dugundji [44], Istratescu [64], Khamsi and Kirk [87] and Singh et al. [141] are of special recommendation.

As usual this chapter is elementary in nature in which we collect the preliminary concepts, basic definitions and those results which are utmost required in course of onward discussion.

1.2. SOME CLASSICAL FIXED POINT THEOREMS

In this section, we discuss some classical fixed point theorems, especially the Brouwer [14] fixed point theorem and Banach's contraction principle [9] and some of their extensions. Though Banach's contraction principle [9] is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis.

As mentioned earlier, Brouwer [14] proved the earliest fixed point theorem which runs as follows:

THEOREM 1.2.1. Let C be the closed unit ball in 3i" and T : C ^ C a continuous mapping. Then *T* has a fixed in *C*.

The Brouwer fixed point theorem is one of the most well known and useful theorems in topology. Since the theorem and its extensions are powerful tools in showing the existence of solutions of many problems in piure and applied mathematics, many researchers have been studying its further extensions and appUcations. An immediate corollary of this theorem on the real Une can be stated in the following way.

COROLLARY 1.2.1. Every continuous self mapping of a closed interval has a fixed point.

Most of the problems of Functional analysis arise in function spaces and sequence spaces and therefore, it is natural to ask if the theorem can be extended to these spaces. Kakutani [79] produced an example to show that Brouwer theorem cannot be extended to infinite dimensional spaces.

EXAMPLE 1.2.1. Let *c* = *{x e l^* -. *x*\\ < 1} be the unit ball in the Hilbert space *e"^*. For each *x* = {xi, X2,X3,...} in C and define a mapping T : C -> C by

Tx = {v^l-||x||2,Xi,X2, . . . ,X„, . . .}.

Then ||Tx|| = 1 and *T* is continuous. But *T* admits no fixed point.

In 1930, Brouwer [14] theorem was extended to infinite dimensional spaces by Schauder [134] which runs follows:

THEOREM 1.2.2. Every continuous self mapping of a compact convex subset of a Banach space has at least one fixed point.

The compactness condition on subset is a stronger one. Many problems in analysis do not have a compact setting. It is natural to modify the theorem by relaxing the condition of compactness. Schauder [134] also proved a theorem for a compact map which is known as second form of Theorem 1.2.2. Before stating the theorem, we need the following definition.

DEFINITION 1.2.2. A self mapping T of a Banach space *X* is called a compact mapping (or completely continuous) if *T* is continuous and *T* maps bounded set to precompact set.

REMARK 1.2.1. A compact mapping is always continuous but converse need not be true. For example, an identity mapping defined on an infinite dimensional normed space is continuous but not compact.

The following is another form of Schauder [134] fixed point theorem.

THEOREM 1.2.3. Every compact self mapping of a closed bounded convex subset of a Banach space has at least one fixed point.

This theorem is of great importance in the numerical treatment of equations in analysis.

In 1935, Tychonoff [151] extended Brouwer's result to a compact convex subset of a locally convex topological vector space.

THEOREM 1.2.4. A continuous self mapping of a nonempty compact convex subset of a locally convex topological vector space has a fixed point.

DEFINITION 1.2.3. A self mapping T of a metric space *{X, d)* is said to be Lipschitzian if *d{Tx,Ty) < k d{x,y)* for all *x,y e X* and A; > 0. T is said to be contraction on X if fc € [0,1) and nonexpansive if A; = 1.

In 1922, Polish mathematician S. Banach [9] gave the most natural and significant result of metric fixed point theory which is popularly known as Banach contrax^tion principle. Banach contraction principle for contreiction mappings asserts that

THEOREM 1.2.5. Every self contraction of a complete metric space has a unique fixed point.

Due to simplicity and usefulness of this celebrated theorem, it has become a very popular source of existence and uniqueness theorems in different branches of mathematical analysis. This theorem provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplines.

In last eight decades, various extentions and generalizations of Banach fixed point theorem have been established. To mention a few, we cite Bryant [16], Edelstein [28], Boyd and Wong [13], Nadler [101], Jungck [70], Grabiec [43] and others. Due to the limitation of the space, it is not possible to mention all the existing contraction conditions. However, we enlist some contraction conditions which are very useful. For a comprehensive survey, one can referred to Rhoades [126,128].

(ag) Edelstein [28]: *d{Tx,Ty) < d{x,y).*

(aio) Kannan [84]: *d{Tx,Ty) < k[d{x,Tx)+d{y,Ty)]* with fc < i.

(an) Boyd and Wong [13]: *d{Tx, Ty) < (l){d{x, y)),* where *(i>:R-^R, (pit) < t* for *t>0* together with a semicontinuity condition of *(f).*

(ai2) Ciric [24]: *d{Tx,Ty) < kmax{d{x,y),d{x,Tx),d{y,Ty),d{x,Ty),d{y,Tx)}* for all *x,y e X* and *k e* [0,1).

(an) Bianchini [10]: *d{Tx,Ty) < kmax{d{x,Tx),d(y,Ty)}* with /c < 1.

(au) Chatterjea [19]: *d{Tx,Ty) < k[d{x,Ty) + d{y,Tx)]* with fc < i.

(ois) Hardy and Rogers [46]:

d{Tx, Ty) < a [d(x, *Tx)* + d(y, *Ty)]* + 6 [rf(x, *Ty)* + d(y, To;)] + c d(x, *y)*
for all *x,y e X* with *x^y,a,b,c>0* such that 2a + 26 + c < 1.

(cie) Khan et al. [90]:

<;&(rf(rx, *Ty)) < a d{x,y)<i>{d{x,y)) + b d{x,y)[Hd{x,Tx)) + cl>{d{y,Ty))]*

+ *c d{x,y)* min{^(rf(x, *Ty)) + <t>{d{y, Tx))}*

for all *x,y £ X* with *x ^y, a,b* and c are three decreasing functions from i2+ - {0} -> [0,1) such that *a{t) + 2b{t) + c{t)* < 1 for every *t* > 0, where *(f): R^ -^ R^ is* an increasing and continuous function satisfying 0(f) = 0 if and only if t = 0.

In endeavour to generalize Banach contraction principle for a pair of mappings, Eldon Dyre (1954), A. Shields (1955) and Lester Dubins (1956) almost simultaneously proposed the following conjecture.

Let S and T be two continuous self mappings of a unit interval which commute under functional composition. Do they have a common fixed point ?

This conjecture was settled in negative by Boyce [12] and Huneke [49] independentally and the answer was given by constructing a pair of commuting mapping with no common fixed point employing a limiting process. Later on, in 1969 Huneke [49] also furnished two precise counter examples to this conjecture on commuting continuous mappings of the closed unit interval. Thus in order to coin a common fixed point theorem, one is required to impose extra conditions either on the space or on the mappings under consideration which is evident in all existing common fixed point theorems. Jungck [70] was perhaps the first mathematician who generalized Banaxih contraction principle by proving a common fixed point theorem whose concrete statement is given in Chapter 4.

In recent years, the theorem due to Jungck [70] has been generalized by many researchers in various ways and by now there exists extensive literature around this theorem. To mention a few, one may cite Fisher [37], Jungck [72,73], Jungck et al. [76], Pathak et al. [119], Singh and Mishra [144], Jungck and Pathak [77], Imdad et al. [63], Meade and Singh [96], Pathak and Khan [120], Popa [122,123], Imdad and Kumar [62] and many others. In Chapters 4 and 5, we also generalize Jungck [70] theorem without any condition on the containment of ranges of the involved mappings.

1.3. MULTI-VALUED MAPPINGS

In this section, we describe some related concepts of multi-valued mappings, definitions and related results along with illustrative examples which will be utilized in our subsequent discussion.

Let *X* and *Y* be two sets. A multi-valued mapping *F* from *XtoY*, denoted by F : X —» F, is a subset *F C. X xY.* The inverse of *F* is the multi-valued mapping F~^ *: Y* -^ *X* defined by *{y,x)* G *F~^* if and only if (x,y) G *F.* The values of *F* are the sets *F{x)* = {y G F : (x,y) G F}; the fibres of *F* are the sets *F~\y)* = *[xeX : {x,y)* G *F}* for *y eY.* Thus, the value of F"^ for y G F is the fibre *F-\y).*

For ^ C X, the set

F(A) = *U^eAFix)* - {y G y : *F'^y)* n A / 0}

is called the image of *A* under F; for *B CY^* the set

F-\B) = *UyeBF-'iy)* = *{xeX: F{x)* n B ^ 0}

the image of *B* under F~\ is called the inverse image of *B* under F.

DEFINITION 1.3.1. Let X and y be topological spaces. A multi-valued mapping F : X -^ y is called

(017) upper semicontinuous if the inverse image of a closed set is closed, and

(ois) lower semicontinuous if the inverse image of an open set is open.

Recall that a multi-valued mapping *F* is called continuous if it is upper as well as lower semicontinuous.

EXAMPLE 1.3.1. The mapping F : 3? —) Si defined by

' {0}, ifx^O

Fix) = {

J-1,1], ifx = 0

is upper semicontinuous but not lower semicontinuous.

EXAMPLE 1.3.2. The mapping *F : dt —i-* ^ defined by

' {0}, ifa; = 0

Fix)={

J-1,1], ifx^O

is lower semicontinuous but not upper semicontinuous.

Let *iX,d)* be a metric space. Then, following Nadler [101], we recall

(aig) *CBiX) = {Ae2^ : Ais* nonempty closed and bounded set},

(020) *CiX) = {A e2^ : Ais* nonempty compact set}.

(021) For nonempty subsets *A,B of X* and *x e X*

d(x, *A)* = inf {d(x, *a) : a G A},*

diA, B) = inf {rf(a, *b) : a e A, b e B},*

6iA,B) = sup *{dia,b) : a e A, b e B}* and

HiA, B) = max {{sup d(a, *B) : a £ A},* {sup d(A, *b) : b e B}}.*

Notice that *diA,B) < HiA,B) < 6iA,B).* Also

(022) *SiA,* B) = 0 if and only *iiA = B = {a}*

(023) *5iA,B) = 5iB,A)*

(024) *SiA,B)<5iA,C) + 5iC,B)*

DEFINITION 1.3.2[101]. Let K be a nonempty closed subset of metric space *{X, d).* *A* mapping *T : K -¥ CB{X)* is said to be continuous at Xo G iC if for any e > 0, there exists a 5 > 0 such that *H{Tx, Txo)* < e, whenever *d{x, Xo)* < *5.UT* is continuous at every point of *K,* then *T* is continuous on *K.*

LEMMA 1.3.1(101]. Let *A, Be CB{X)* and fc > 1. Then for each *a e A,* there exists a point *b e B* such that d(a,6) < *kH{A, B).*

LEMMA 1.3.2(101]. *Let A,B e CB{X)* and a e A, then for any positive number *q <l* there exists *h = b{a)* in *B* such that *q d{a, b)* < *H{A, B).*

Alternately, if a G ^ and e > 0 then using above lemma, one can always find *beB* such that rf(a,6) < *H{A,B)* + e.

Fixed point theory for multi-valued mappings was originally initiated by von Neumann in the study of game theory. Fixed point theorems for multi-valued mappings are quite useful in control theory and have been frequently used in solving the problems in Economics and Game theory.

The study of fixed point theorems for multi-valued mappings in finite dimensional spaces was initiated by Kakutani [79]. His result was indeed a step forward in an endeavor to extend Brouwer's fixed point theorem to multi-valued mappings which was later extended to infinite dimensional Banach spaces by Bohnenblust and KarUn [11] whereas, to locally convex spaces by Fan [31]. The development of geometric fixed point theory for multi-valued mappings was initiated by Nadler [101] which has been further improved and enriched by Markin [95], Assad and Kirk [5], Browder [15], Himmelberg [48], Jafari and Sehgal [68] and several others.

DEFINITION 1.3.3. A multi-valued mapping T : X -> X is said to have a fixed point if the point belongs to its image set (i.e. *z eTz* for some *z e X).*

DEFINITION 1.3.4. Let X be a metric space and *CB(X)* the family of nonempty bounded closed subsets of *X.* *A* multi-valued mapping *T : X -^ CB{X)* is called a Lipschitz mapping with Lipschitz constant A; > 0, if *H{Tx, Ty)* < *kd{x, y),* for all *x,y E X.* r is called nonexpansive if /c = 1 and a set-valued contraction if fc < 1.

In 1969, Nadler [101] proved the following result as a multi-valued analogue of Banach contraction principle.

THEOREM 1.3.1. Every multi-valued contraction on a complete metric space has a fixed point.

1.4. CONVEXITY IN METRIC SPACES

In 1928, Menger [97] introduced the following concept of metric convexity.

DEFINITION 1.4.1, A metric space *{X,d)* is said to be metrically convex if for any *x,y e X* with *x ^y,* there exists a point *z e X, x^ z^y* such that

d{x, z) + d{z, y) = d{x, y)

i.e. *z* lies between *x* and *y* which is generally denoted by the symbol *(xzy).*

Here we quote two useful lemmas which are relevant to present exposition.

LEMMA 1.4.1 [97]. Let *(X, d)* be a complete metric space with *x,y e X,x ^y,* and let

B{x,y) = {zeX : {xzy)},

S = S{x,y, A) = {2 e 5(2;, *y) : d{x, z)<X}U{x},* 0 < A < *d{x, y).*

Then there exists a point *zx^ X* such that

(025) *zx* G *S{x,y,X),*

{0,26) u G *B{x,y)* and *(xzxu)* implies *d{x,u)* > A.

LEMMA 1.4.2 [97]. Let *X* be complete and convex, *x,y £ X, x ^y* and suppose 0 < A < *d{x,y).* Then there exists *z' e X* such that *(xz'y)* and *d{x,z')* — A.

Using Lemmas 1.4.1 and 1.4.2, Menger [97] proved the following fundamental theorem of metric convexity.

THEOREM 1.4.1. if *{X,d)* is a complete and metrically convex metric space, then any two points *x,y ^ X* are joined by a metric segment, i.e., there exists an isometry *(f):* [0, *d{x, y)] ->• X* with (/>(0) = *x* and *4>{d{x, y)) = y.*

In 1972, Assad and Kirk [5] initiated the study of fixed point theorems for nonself mappings in metrically convex spaces. The following lemma and theorem are of frequent use in course of our subsequent discussion.

LEMMA 1.4.3[5]. Let A" be a nonempty closed subset of a metrically convex metric space *X.lix E K* and *y* ^ *K,* then there exists a point *z* € *SK* (the boundary of *K)* such that

$$d\{x,z) + d\{z,y) = d\{x,y).$$

THEOREM 1.4.2[5]. Let X be a complete and metrically convex metric space, *K he a,* nonempty closed subset of *X* and T be a contraction mapping from *K* into *CB{X). UTxCK* for each *xeSK* then there exists *ZEK* such that *z* G *T{z).*

For the sake of completeness, we also include the definition of the boundary of a set in a metric space.

DEFINITION 1.4.2. Let *K he a* nonempty subset of a metric space *{X,d).* A point X 6 A" is said to be the boundary point of /C if x is neither an interior point of *K* nor an exterior point of *K.* (In other words, x 6 X is said to be a boundary point of *K* if every open sphere centered on x intersects both *K* and *{X — K)).* The boundary of *K* will be denoted by *5K.*

In recent years. Theorem 1.4.2 (due to Assad and Kirk [5]) has been generalized and improved in various ways and by now there exist considerable Uterature around this result. To mention a few, we cite Rhoades [127], Imdad and Khan [54,55], Assad [6,7], Imdad et al. [56], Khan et al. [91], Abdalla and Zaheer [8] and others. Apart from the results on multi-valued or hybrid pairs some interesting fixed point theorems for single-valued mappings are also proved which includes Rhoades [127], Imdad and Kumar [62] and others. The results on single-valued mappings of Chapter 6 revolve around the results contained in Assad [7], Imdad and Khan [54], Imdad et al. [56], Khan and Bharadwaj [88], Khan et al. [91] and some others.

On the other hand, Takahashi [147] introduced another concept of convex metric spaces different from the one given by Menger [97]. Takahashi [147] also investigated some properties of such spaces and proved related results besides some fixed point theorems for nonexpansive mappings.

DEFINITION 1.4.3. A metric space *{X,d)* is said to be convex metric space if *X* has a convex structure *W{x,y].)* (i.e. a mapping *W : X x X x [0,1] -^ X)* and satisfying

$$d\{u, W\{x, y; \text{A})) < Xd\{u, x) + \{1 - X)d\{u, y)$$

for all *u,x,y e X* and for all A e [0,1].

A Banach space and its all convex subsets are convex metric spaces.

EXAMPLE 1.4.1. A linear space *X* which is also a metric space with the following properties:

(027) for *x,yGX, d{x,y)* = *d{x* - y,0),

(a28) for *x,y e X* and A G [0,1],

$$d\{Xx + (1 - X)y, 0) < Xd\{x,0) + (1 - X)d\{y, 0).$$

Then X is a convex metric space.

DEFINITION 1.4.4. A subset *K of a.* convex metric space *X* is said to be convex if *W{x,y;X)eK* for all *x,y e K and X e* [0,1].

Takahashi [147] also proved the following natural propositions.

PROPOSITION 1.4.1. Let *{X,d)* be a convex metric space. For *x,y E X* and AG [0,1],

$$d\{x, y) = d\{x, W\{x, y; \text{A})) + d\{W\{x, y\backslash X),y).$$

PROPOSITION 1.4.2. Let *{K^* : a G A} be a family of convex subsets of a convex metric space *{X,d),* then *Da^AKa* is also convex.

PROPOSITION 1.4.3. The open sphere *S{x,r)* and closed sphere *S[x,r]* of a convex metric space *{X, d)* are convex subsets of *X.*

Finally, we conclude this section by recalling some elementary definitions which will also be required in our subsequent discussion.

DEFINITION 1.4.5. A subset /C of a linear space *X* is said to be convex if Aa; + (1 - \)*y e K iov bX\. x,y e K* and A e [0,1].

DEFINITION 1.4.6. A subset K" of a linear space *X* is said to be starshaped if there exists at least one point *p e K* such that for each *x E K* and A e [0,1], Aa; + (1 - *X)p* G *K.*

Clearly, a starshaped set is convex but not conversely.

DEFINITION 1.4.7. Let X be a linear space and T be a self mappings on *X.* Then *T* is said to be affine if

$$T\{Xix + X2y) = XiTx + \mathrm{AjTy}$$

for all *x,y E X* and Ai + A2 = 1.

DEFINITION 1.4.8. Let X be a normed Unear space and *Khea.* nonempty subset of *X.* A mapping *T : K* -^ *X is* said to be demiclosed if {xn} ^ *K,* G *K* and *Txn* -> y G X implies *Tx* = *y.*

In what follows, —^ (->) denotes weak (strong) convergence.

1.5. FUZZY METRIC SPACES

It proved a tiurning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [156] which laid the foundation of fuzzy mathematics. Since then many authors have extensively developed the theory of fuzzy sets and applications. Especially, Deng [26], Erceg [29], Kaleva and Seikkala [81], Kramosil and Michalek [92] and Xia and Guo [155] have introduced the concepts of fuzzy metric spaces in different ways. First ever attempt to define fuzzy metric spaces is due to Kramosil and Michalek [155]. In 1984, Kaleva and Seikkala [92] also defined fuzzy metric spaces in a different way.

DEFINITION 1.5.1. Let X be a nonempty set, *d* a mapping from *X xX* into *G,* the set of nonnegative, upper semicontinuous, normal, convex fuzzy numbers and let the mappings L, *R* : [0,1] x [0,1] -4 [0,1] be symmetric, nondecreasing in both arguments and satisfy L(0,0) = 0 and il(l, 1) = 1. Denote the a-level set of *d{x, y)*

$$[d\{x,y)]a \;=\; [K\{x,y),Pa\{x,y)]$$

for all a;,y G *X,* 0 < *a* < 1. The quadruple *{X,d,L,R)* is called a fuzzy metric space and *d* a fuzzy metric, if

(029) *d{x,* t/) = 0 if and only if x = y,

(aao) *d{x,y)* = *d(y,x)* for all *x,yeX,*

(aai) for all *x,y E X*

(i) *d{x,y)(s + t)* > *L{d{x,z){s),d{z,y){t))* whenever *s* < *Xi{x,z),* *t* < *Xi{z,y)* and *s* + *t* < Ai(x,y)

(a) *d{x,y){s + t)* < *R{d{x,z){s),d{z,y){t))* whenever *s* > *Xi{x,z),* *t* > Ai(z,y) and *s* + *t*> *Xi{x,y).*

DEFINITION 1.5.2. A sequence {x„} in a fuzzy metric space *{X,d,L,R)* converges to *X* if $\lim\limits_{Tl->00}$ *d{xn, x)* = 0.

DEFINITION 1.5.3. A sequence {x„} in a fuzzy metric space *{X, d, L, R)* is said to be Cauchy if $\underset{Tn,n—>oo}{\mathrm{Hm}}$ *d{xm,* a;„) = 0.

DEFINITION 1.5.4. A fuzzy metric space *{X,d,L,R)* is said to be complete if every Cauchy sequence in *X* converges.

In the continuation, Kaleva [80] studied more properties of fuzzy metric spaces and proved the following completion theorem for fuzzy metric spaces.

THEOREM 1.5.1. Let *{X,d,Min,Max)* be a fuzzy metric space with

$$lim d\{x,y)\{t) = 0 \quad \text{forallx,i/GX} \tag{1.1}$$

Then *{X, d, Min, Max)* has a completion which is unique up to isometry.

In fact, Kaleva [80] proved this theorem using similar steps as adopted in the case of completion theorem for a metric space.

However, Grabiec [43] was the first mathematician who proved Banach contraction principle and Edelstein [28] theorem in fuzzy metric spaces (in sense of

Kramosil and Michalek [92]). He also defined Cauchy sequence in the following way.

DEFINITION 1.5.5. A sequence {x„} in a fuzzy metric space *{X,* M, •) is Cauchy if lim *Mixn+p,* x„, *t)* = *1* for each *t>Q* and p > 0.
n-¥00 ***r*** , .

LEMMA 1.5.1.[43] Let (X,M,*) be a fuzzy metric space, then M(a;,y,.) is nondecreasing for all x, y G X

THEOREM 1.5.2. [43] Let (X, M, •) be a complete fuzzy metric space such that lim *Mix, y,t)* = \ for all *x,y £X.* Let T be a self mapping of *X* satisfying
t—¥00

M{Tx, Ty, kt) > *M{x, y,t)* for all x, y G X, 0 < fc < 1.

Then *T* has a unique fixed point.

THEOREM 1.5.3. [43] Let (X, M,*) be a compact fuzzy metric space and *T* be a self mapping of *X* satisfying

M{Tx, Ty,.). > M(x, *y,.)* for *aWxj^yeX.*

Then *T* has a unique fixed point.

Later, George and Veeramani [39] modified the concept of fuzzy metric spaces defined by Kramosil and Michalek [92] and shown that the topology on this fuzzy metric space is Hausdorff'. In fact, George and Veeramani [39] defined the fuzzy metric spaces as follows:

DEFINITION 1.5.6. The triplet *{X,M,*)* is said to be a fuzzy metric space if *X* is an arbitrary set, * is a continuous i-norm and M is a fuzzy set on *X^* x (0, oo) satisfying the following conditions:

(032) *M(x,y,t)* > 0 ,

(033) *M{x,y,t)* = *liSx* = *y,*

(034) *M{x,y,t)* = *M{y,x,t),*

(035) M(x, *y, t)* • M(y, *z, s)* < *M(x, z, t* + *s),*

(036) M(x,*y,.)* : (0,00) -^ [0,1] is continuous, for all *x,y,z e X* and *t,s* > *0.*

EXAMPLE 1.5.1. Let $X = \Re$. Define $a * b = ab$ and $M(x, y, t) = e^{\frac{|x-y|}{t}}$ for all $x,y \in X$ and $t>0$. Then $(X,M,*)$ is a fuzzy metric space.

EXAMPLE 1.5.2. Let $X = N$. Define $a*b = ab$ and for all $t > 0$

$$M(x,y,t) = \begin{cases} x/y, & \text{if } x \le y \\ y/x, & \text{if } y \le x. \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space.

Here it may be pointed out that the above function M is not a fuzzy metric with the t-norm defined as $a * b = \min\{a, b\}$.

DEFINITION 1.5.7. Let $(X,M,*)$ be a fuzzy metric space. One can define open ball $B(x,r,t)$ as well as closed ball $B[x,r,t]$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$ as

$$B(x,r,t) = \{y \in X: M(x,y,t) > 1 - r\},$$

$$B[x,r,t] = \{y \in X: M(x,y,t) \geq 1 - r\}.$$

THEOREM 1.5.4. Every open (closed) ball is an open (a closed) set.

George and Veeramani [34] also proved that every fuzzy metric space is Hausdorff and topology induced by the metric d and topology induced by the standard fuzzy metric M are the same besides proving Baire's Category theorem in fuzzy metric spaces. They also observed that the definition of Cauchy sequence given by Grabiec [43] is weak. In this regard, they furnished an illustrative example which contradicts the completeness of $\Re$. Keeping this view point, they redefined the Cauchy sequences as follows:

DEFINITION 1.5.8. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is Cauchy iff for each $\epsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n,m \geq n_0$.

In recent years, many authors utilized fuzzy metric spaces to prove their results on fixed and common fixed point theorems. To mention a few, one can cite

[20,22,32,52,108]. Jungck [72] and Pant [105] introduced the notions of compatibility and reciprocal continuity, respectively in metric spaces. Most recent, Pant and Jha [108] extended the reciprocal continuity to fuzzy setting and proved some common fixed point theorems with a related result. We opt to present the following definitions to maice our presentation as self-contained as possible.

DEFINITION 1.5.9. A pair (S, T) of self mappings of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 1.5.10. A pair (S, T) of self mappings of a fuzzy metric space $(X, M, *)$ is said to be reciprocally continuous if $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

If S and T are both continuous then they axe obviously reciprocally continuous but reverse imphcation does not hold. Pant and Jha [108] also observed that if the pair (S, T) is compatible and one component is continuous, then the pair (S, T) is reciprocally continuous. We have also observed that if the pair of mappings is compatible and reciprocally continuous, then the mappings have at least one coincidence point even in absence of any contraction condition. We have also extended some notions of weak commutativity to fuzzy metric spaces whose details are available in Chapter 7.

Chapter 2

COINCIDENCE AND FIXED POINTS OF SELF MAPPINGS UNDER STRICT CONTRACTIONS

2.1. INTRODUCTION

While considering Lipschitzian mappings, a natural question arises whether it is possible to weaken contraction assiunption a little bit in Banach contraction principle and still ensures the existence of fixed point. In general the answer to this question is no. In order to substantiate this view point, the following interesting example can be adopted which is available in Khamsi and Kirk [87].

EXAMPLE 2.1.1. Let C[0, l] be the complete metric space of real valued continuous functions defined on [0,1] under supremum metric and consider the closed subspace *X* of C[0,1] consisting of those functions / G C[0,1] satisfying /(I) = 1. Since X is a closed subspace of C[0,1], hence *X* is also complete. Now, define *T : X ^ X by Tf{t) = tf{t),* V t 6 [0,1]. Then one can easily verify that *d{Tf, Tg) < d{f, g)* whenever *f ^ghntT\\B&* no fixed point *asTf = f =^ tf = f => f{t) = 0, y t G[0,* 1). On the other hand, /(I) = 1 which contradicts the continuity of / and so *T* cannot have a fixed point in *X.* Here one may note that T is a contractive mapping on *X.* Let us recall that a self mapping T on a metric space *{X, d)* is said to be contractive (cf. [28]) if *d{Tx,Ty) < d{x,y), \/ x^yeX.*

In view of above example, a contractive condition does not ensure the existence of fixed points unless the underlying metric space is compact (cf. [28]) or the contractive conditions are replaced by relatively stronger conditions such as Banach type contraction condition (cf. [72,75,109]) or Meir-Keeler type condition (cf. [66,74,103,104,107]). Recently, Aamri and Moutawakil [1] (also Pant and Pant [109]) obtained some relatively more general common fixed point theorems for strict contractive conditions in complete metric spaces. In fact, Aamri and Moutawakil [1] proved the following result in metric spaces.

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THEOREM 2.1.1. Let A and *S* be two weakly compatible self mappings of a metric space *{X, d)* such that

(61) the pair *{A, S)* satisfies property *(E.A),*

(62) for *allx^y eX*

diAx, Ay) < max |d(5x, *Sy),* ^(-^x. *Sx)* + *djAy, Sy)* ^ *djAy, Sx)* + *d(Ax, Sy)* | ^

(63) *A{X)CS{X).*

If *A{X)* or ^(X) is a complete subspace of *X,* then *A* and 5 have a unique common fixed point.

The main objective of this chapter is to obtain some results on coincidence and common fixed points without continuity requirements satisfying a slightly more general contractive condition which also admits a nonmetric distance function *d* with the property that sequence {a:„} converges to *x* if and only if *d(xn,x)* -^ 0. We choose symmetric spaces as well as semi-metric spaces as our underlying spaces. In process, some recent results due to Aamri and Moutawakil [1, Theorem 1], Pant and Pant [109, Theorems 2.1 and 2.3] and some others extended to symmetric (semi-metric) spaces. We also observe that results contained in [1,109] remain true (upto coincidence points) even in symmetric (semi-metric) spaces for a slightly general contractive condition besides the possibility of sharpening other conditions as well. In the end of the chapter, we derive some related results besides furnishing illustrative examples which establish the utility of the results proved in this chapter.

Before presenting our results, let us recall the relevant definitions and motivations.

DEFINITION 2.1.1. A symmetric on a set X is a function *d : X x X* —^ *[0,* oo) such that for all *x,y E X*

(64) *d{x,y)* = *0^x* = *y,*

(65) *d{x,y)* = *d(y,x).*

If d is symmetric on the set X, then for x e A" and c > 0, we write $B\{x, e) = \{y e X : d\{x, y) < e\}$. k topology r(d) on X is given by (/ G T(rf) if and only if for each X e U, $B\{x, e) C U$ for some e > 0. A set 5 C X is a neighbovirhood of x G X iff there exists Ue r(d) such that $x e U C S$. A symmetric d is a semi-metric if for each $X e X$ and for each e > 0, S(x, e) is a neighbourhood of x in the topology $T\{d)$.

DEFINITION 2.1.2. A semi-metric space X is a topological space whose topology r(d) on X is induced by semi-metric d. In what follows symmetric space as well as semi-metric space will be denoted by $\{X,d)$.

The distinction between a symmetric and a semi-metric is evident as one can easily construct a symmetric d such that $B\{x$, e) need not be a neighbourhood of X in r(d). For a symmetric *don* X the following two axioms were given by Wilson [154].

(be) Given {x„}, x and y in X, d(x„,x) -> 0 and $d\{xn,y)$ -> 0 imply that $x -y$.

(67) Given {xn},{y„} and an x in X, $d\{x„,x)$ -> 0 and d(x„,y„) ->• 0 imply that $d\{yn,x)$ -> 0.

Here it may be noted that for a semi-metric d if r(d) is Hausdorff, then *(be)* holds.

Now we state some weak commutativity conditions from the existing literature which are relevant in the present context and can be naturally adopted to the setting of symmetric (semi-metric) spaces.

DEFINITION 2.1.3. [102] A pair $\{A, S)$ of self mappings defined on a symmetric (semi-metric) space $\{X, d)$ is said to be /?-weakly commuting if there exists some real number $R > 0$ such that

$$d\{ASx, SAx) < R.d\{Ax, Sx)$$

for all $X E X$, whereas the pair $\{A, S)$ is said to be pointwise /?-weakly commuting if for given $x G X$ there exists $R > 0$ such that

$$d\{ASx,SAx) < R.d\{Ax,Sx).$$

Here it may be pointed out that on the set of coincidence points i?-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points of contractive type mappings.

DEFINITION 2.1.4. [72] A pair *{A,S)* of self mappings defined on a symmetric (semi-metric) space *{X, d)* is said to be compatible if

$$\lim_{n \longrightarrow oo} d\{ASxn,SAxn) = 0$$

whenever *{xn}* is a sequence in *X* such that $\lim_{n \longrightarrow oo} Axn = \lim_{n \longrightarrow oo} Sxn = t \wedge X.$

Here it may be noted that pointwise i?-weakly commuting mappings need not be compatible.

DEFINITION 2.1,5.[1] A pair *{A, S)* of self mappings defined on a symmetric (semi-metric) space *{X, d)* is said to enjoy property *(E.A)* if there exists a sequence *{xn}* such that $\lim_{n—Kx>} Ax_{„} = \lim_{n \longrightarrow oo} 5x_{„} = t$ for some t G X.

Clearly compatible pairs satisfy property *(E.A)*.

DEFINITION 2.1.6. A pair *{A,S)* of self mappings defined on a nonempty set *X* is said to be weakly compatible if *Ax* = *Sx* for some x G X implies *ASx* — *SAx*.

The notion of weak compatibility is also defined by some other authors under the different names (e.g. coincidentally commuting (cf. [27]) and partially commuting mappings (cf. [133])). Here it may be pointed out that Jungck [74] defined this notion in metric space. But the notion of weak compatible mappings never involves the metric of underlying space.

The organization of this chapter is as follows: In Section 2.1, we have already collected the relevant definitions and results which serve as a backup material to the contents of this chapter. In Section 2.2, we present some results on coincidence and common fixed points besides establishing the utility of our results with the aid of an illustrative example. In the last section, we prove results on coincidence and common fixed points for two pairs of self mappings satisfying a functional inequaUty control by a suitable control function. The genuineness of oiu: results over the relevant ones is supported by an illustrative example.

2.2. RESULTS FOR A PAIR OF MAPPINGS

Our main result runs as follows:

THEOREM 2.2.1. Let *{X,d)* be a symmetric (semi-metric) space that enjoys *(be)* (the Hausdorffness of *T{d)).* Let *A* and *S* be two self mappings of *X* such that

(bs) the pair *{A,S)* enjoys the property (£^.^4),

(bg) *(oT aUx^y eX*

$$\Big\{ \quad k \quad k \quad)$$

diSx, Sy), -[d{Ax, Sx) + diAy, Sy)], -[d{Ay, Sx) + d{Ax, Sy)] |,

(2.1)

1 < fc < 2. If *S{X)* is a d-closed (T(d)-closed) subset of *X,* then *A* and 5 have a point of coincidence.

PROOF. Firstly, one needs to note that a sequence {x„} in a semi-metric space *{X, d)* converges to a point *x* in r(d) iflF *d{xn, x)* $\longrightarrow$ 0. To substantiate this, suppose and let e > 0. Since *B{x,* e) is a neighbourhood of *x* there exists *U* 6 T(d) such that *X e U C B{x, e).* Since x„ -> x there is an m G iV (the natural number) such that *Xn G U C B{x,e)* for n > m so *d{xn,x)* < e for n > m, i.e., d(a;„,x) -> 0. The converse part is obvious in view of the definition of *T{d).*

Now in view of (ftg), there must exist a sequence {x^} in *X* with t G X such that lim *Axn* — hm 5x„ = *t* ^ *X.*
n—¥oo n—>oo

As *S{X)* is d-closed, every convergent sequence of points of *S{X)* has a limit in *S{X),* therefore lim *Sxn* = *t* = *Sa*= lim *Axn* for some *a €: X* which in tmrn yields
n—>oo n-»oo

that *t* = *Sa* £ *S{X).* Now we assert that *Sa* = *Aa.* If it is not so then in view of (2.1), one gets

$$d\{Axn, Aa) < \max \overset{f}{<} d(5x„, Sa), \overset{k}{-}[rf(5x„, Axn) + d\{Aa, Sa)], \overset{k}{-}[d\{Sa, Axn)$$

$$+d\{Sxn,Aa)]\backslash$$

which on letting n -4 oo, reduces to

$$\left\{ \overset{k}{-}d\{Aa,Sa), \overset{k}{-}d\{Sa,Aa) \right.\overset{']}{>} < d\{Sa,Aa)$$

yielding thereby *Sa* = *Aa,* which shows that a is a point of coincidence for *A* and 5. The same proof works for the alternate statement. This completes the proof.

Since a pair of noncompatible mappings also satisfies the property *(E.A).* Therefore, we have the following result for a pair of noncompatible mappings.

COROLLARY 2.2.1. Let *{X,d)* be a symmetric (semi-metric) space that enjoys *(be)* (the Hausdorffness of r(d)). Let *A* and *S* be two self mappings of *X* such that

(610) the pair *{A, S)* is noncompatible,

(611) for *allx^y eX*

$$d\{Ax, Ay) < \max ld\{Sx, Sy), '\hat{}[d\{Ax, Sx) + d\{Ay, 5y)], \hat{}[d\{Ay, Sx) + d\{Ax, Sy)] \backslash ,$$

1 < k < 2. If *S{X)* is a d-closed (T(d)-closed) subset of *X,* then *A* and *S* have a point of coincidence.

The following variant of Theorem 2.2.1 also remains true.

THEOREM 2.2.2. Theorem 2.2.1 remains true if d-closedness (r(d)-closedness) of *S{X)* is replaced by d-closedness (r(rf)-closedness) of A(X) along with *A{X)* C *S{X)* retaining the rest of the hypotheses.

PROOF, since *A* and *S* enjoy property *(E.A),* we have $\lim_{n \to oo} Sxn = \lim_{n \to oo} Axn$ — *Aa = t £ X iov* some *a E X as A{X)* is a d-closed subset of *X.* Now due to *A{X)* C *S{X)* one can find some *b E. X* such that *Aa = Sb.* Suppose on contrary that *Aa ^ Ab;* then using (2.1) one obtains

$$d\{Axn, Ab) < \max \{d(5x„, Sb), -[d\{Axn, 5x„) + d\{Ab, Sb)], \underset{2'}{-}[d\{Ab, Sxn)$$

$$+d\{AXn,Sb)]\}$$

which on letting $n \to \infty$, reduces to

$$\left\{ \; k \qquad\qquad k \qquad\qquad \wedge \right.$$

$$-d\{Ab,Aa),-d\{Ab,Aa) \quad > \quad < \quad d\{Aa,Ab)$$

yielding thereby $Aa = Ab = Sb$ as desired.

Like Pant and Pant [109], Theorems 2.2.1 and 2.2.2 ensure common fixed point instead of point of coincidence if contractive condition (2.1) is replaced by a slightly weaker condition. In this regard we have the following.

THEOREM 2.2.3. in the setting of Theorems 2.2.1 and 2.2.2, A and S have a unique common fixed point provided A and S are weakly compatible and contraction condition (2.1) is replaced by the following: for *all* $x \wedge y \in X$

$$d\{Ax, Ay) < \max \mid d\{Sx,Sy), -[d\{Ax,Sx) + d\{Ay,Sy)], -[d\{Ay, Sx) + d\{Ax, Sy)] \; \mathrm{I}, \tag{2.2}$$

where $1 < A; < 2$.

PROOF, in view of Theorems 2.2.1 and 2.2.2, A and S have a point of coincidence, say a i.e., $Aa = Sa$. Now due to weak compatibihty of the pair $\{A, S)$, one can write $AAa = ASa = SAa = SSa$. If $AAa \wedge Aa$, then (2.2) implies

$$d\{Aa, AAa) < \max \overset{f}{<} d\{Sa, SAa), \overset{k}{-}[d\{Aa, Sa) + d\{AAa, SAa)], \overset{1}{-}[d\{AAa, Sa)$$

$$+d(Aa,SAa)]\backslash \quad < \quad d\{Aa,AAa),$$

a contradiction. Hence $Aa = AAa = ASa = SAa$, which shows that Aa is a common fixed point of A and S. Uniqueness of the common fixed point follows easily.

REMARK 2.2.1. Theorem 2.2.3 generalizes relevant fixed point theorems due to Aamri and Moutawakil [1] and Pant and Pant [109].

REMARK 2.2.2. Theorems 2.2.1 and 2.2.2 remain true, if one replaces contractive condition (2.1), i.e.

$$diAx, Ay) < \max \mid d\{Sx, Sy), -[d\{Ax, Sx) + d\{Ay, Sy)], -[d\{Ay, Sx) + d\{Ax, Sy)] \; \backslash \;,$$

where 1 < A; < 2 by

$$d\{Ax, Ay) < \max\{d\{Sx, Sy), k[d\{Ax,Sx) + d\{Ay, Sy)],k[d\{Ay, Sx) + d\{Ax, Sy)]\},$$

where 0 < fc < 1.

We now furnish an example to demonstrate the validity of the hypotheses and degree of generaUty of our results over earUer ones especially those contained in [1,109]. Our example presents a nonmetric setting satisfying the hypotheses of Theorem 2.2.3 which in turn establishes the genuineness of our results over all relevant metrical fixed point theorems.

EXAMPLE 2.2.1. Consider X = [0,1] equipped with the symmetric $d\{x, y)$ = (x - $y)'$^. Define $A,S : X$ -^ X *as* follows:

$$S\{x) = \{ \begin{array}{ll} 1 - x, & \text{if } 0 < x < \wedge \\ 0, & \text{if } i < x < 1, \end{array} \qquad A\{x) = \{ \begin{array}{ll} I, & \text{ii } 0 < X < \wedge \\ |, & \text{if } i < \wedge \wedge 1 - \end{array}$$

Clearly $S\{X)$ = {0}U[|, 1] is d-closed in X. The pair $\{A,S)$ enjoys the property *(E.A)* as for the sequence {| - ^} C [0,1], we have

$$\lim_{n\text{-foo}} A(1 \ \text{V2} \ --)_{nJ} = \lim_{n\to\infty} S(l--)_{\backslash 2 \ 71/} = le_{2} \ [0,1].$$

By a routine calculation one can show that the contractive condition (2.1) holds for every x ^ y e X. Also notice that the $A\{\wedge)$ = $S\{\wedge)$ => $AS\{\wedge)$ = $SA\{\backslash)$. Since the topology induced by d is usual on [0,1], it will be Hausdorff and therefore condition *(be)* is naturally satisfied. Thus all the conditions of Theorem 2.2.3 are satisfied and ^ is the unique common fixed point of A and S. Here, one needs to note that d is not a metric as rf(0, l) = l > i + i = d(0, i) + d(|, 1). Thus all the available metrical common fixed point theorems cannot be used in the context of this example. Notice that both the mappings A and S are discontinuous at the unique common fixed point ^.

Here it may be observed that Example 2.2.1 also satisfies the requirements of Theorems 2.2.1 and 2.2.2 as $A\{X)$ = {^ |} C {0} U [|, 1] = $S\{X)$ and $A\{X)$ is d-closed. Finally, it may be mentioned that condition (65) is also crucial as this ensures the uniqueness of limit to convergent sequences. It is not difficult to find a symmetric which induces non-Hausdorff topology such as Ti-topology which permits the convergence of a sequence to more than one Umit points (e.g., *X* = ^, *d{x,y)* = ijq;i when *x y^ y* and *d{x,x)* — 0 with *Xn* = *n,ne N).*

2.3. RESULTS FOR TWO PAIRS OF MAPPINGS

Our next theorem involves a function *(f):*^'^-^^'^ which satisfies the following conditions:

(612) *(p* is nondecreasing on Jf!'*',

(^13) 0 < *(f){t)* < *t* for each *t* € (0,oo).

THEOREM 2.3.1. Let *A,B,S* and *T* be self mappings of a symmetric (semi-metric) space *{X,d)* that enjoy *(be)* (the Hausdorffness of T(d)). Suppose that

(614) *A{X)cT{X),B{X)cS{X),*

(615) the pair *{B,T)* (or alternatively the pair (A,5)) enjoys the property *(E.A),*

(ftie) the following inequaUty holds:

$$d\{Ax,By)<<l>\{m(x,y)) \tag{2.3}$$

m{x,y) = max *Usx,Ty), ^[d{Ax, Sx)* + *d{By,Ty)], ^[diAx^Ty)* + *d{By, Sx)]* ,

where 1 < A; < 2,

(617) *S{X)* (or alternately *T{X))* is a d-closed (T(rf)-closed) subset of *X.*

Then the pairs *{A,S)* and *{B,T)* have points of coincidence.

PROOF, since the pair (B,T) enjoys the property *(E.A),* therefore there exists a sequence {x„} C *X* such that $\lim_{n\to\infty}$ *Bxn* = $\lim_{n\to\infty}$ Tx„ = *teX.* Since *B{X)* C *S{X)* for *Xn* there exists y„ such that *Bxn* = 5j/„. Thus in all *BXn* -^ *t,Txn* -> *t* and *Syn* -> *t.* Now we assert that *Ayn* —^ *t.* If not, there must exists a subsequence *{Aym}* of *{Ayn},* a positive number *M* and a number 7 > 0 such that for each m > M, we have *d{Ayn, t)* > 7, *d{Ayni, Bxm)* > 7 and

d{Aym, Bxm) < </>(max *^d{Sym,Txm), -[d{Ayn^, Sym)* + d(Bx„^,^x^)],

$$\begin{array}{ll} \boldsymbol{k} & \mathbf{1}\backslash \\ \sim[diAym,Txm) + d\{Bxm, Sym)]j) & \end{array}$$

$$< d\{Ay\wedge, Sym) = d\{Aym, Bxm),$$

a contradiction. Hence *Aym* -> *t.*

Suppose that *S{X)* is d-closed subset of *X* and 5j/„ -> *t,* then one can find a point *u E X* such that *Su — t.* Now we suppose that *Au* ^ *Su.* Then inequality (2.3) imphes

$$d\{Au,Bxn) < (/)(maxU\{Su,Txn),-[d\{Au,Su) + d\{Bxn,Txn)],-[d\{Au,Txn)$$

$$+d(Bx„,5u)]\})$$

which on letting *n-^ oo,* yields

$$d(Au, Su) <4)(-d\{Au, Su) j < d\{Au, Su)$$

a contradiction. Hence *Au* = *Su.* Also *A{X)* C *T{X),* there exists a point *w e X* such that *Au* = *Tw.* We assert that *Tw* = Biu. If not, then using inequality (2.3), one gets

$$d\{Au,Bw) < <i)(ma.xU\{Su,Tw),-[d\{Au,Su)+d\{Bw,Tw)],\sim[d\{Au,Tw)$$

$$+d(Bu;,5u)]\})$$

$$= 0 (-d\{Bw, Au) j < d\{Bw, Au)$$

(as 1 < fc < 2) a contradiction. Hence *Au* = *Su* = *Bw* — *Tw,* which shows that the pairs *{A, S)* and (B, *T)* have a point of coincidence *u* and K; respectively.

The proof is similar if we consider the case when pair *{A, S)* enjoys property *(E.A)* and *T(X)* is rf-closed subset of *X.* Hence it is omitted. This completes the proof.

On the lines of the Theorem 2.2.3, one can have the following in the context of Theorem 2.3.1.

THEOREM 2.3.2. in the setting of Theorem 2.3.1, *A, B, S* and *T* have a unique common fixed point provided one adds the weak compatibility of the pairs *{A, S)* and

(B, T) besides replacing contractive condition (2.3) with a slightly weaker condition: *ioT X ^y e X*

$$d\{Ax,By)<(f>\{m\{x,y)l \qquad (2.4)$$

$$m\{x,y) = \max\ ld\{Sx,Ty), \ \hat{}[d\{Ax,Sx) + d\{By,Ty)], \ \hat{}[d\{Ax,Ty) + d\{By, Sx)]\backslash \ .$$

PROOF, in view of Theorem 2.3.1, one concludes that *Au = Su — Bw = Tw.* Now the weak compatibility of *{A, S)* implies that *ASu = SAu* and *AAu — ASu = SAu = SSu.* Suppose that *Au ^ AAu;* then using (2.4), one gets

$$d\{Au, AAu) = d\{AAu,Bw) < \overset{/}{M} \ \overset{\mathbf{f}}{\max} U\{SAu,Tw), \overset{k}{-}[d\{AAu,SAu) + d\{Bw,Tw)],$$

$$\mathrm{i}\,[d\{AAu, Tw) + d\{Bw, SAu)]\}\ )$$

$$< d\{Au, AAu)$$

a contradiction. Thus *Au = AAu = SAu,* then *Au* is the common fixed point of the mappings *A* and *S.*

Since the pair *{B, T)* is also weakly compatible, hence *BBw — BTw = TBw = TTw.* Suppose that *Bw ^ BBw;* then in view of (2.4), one gets

$$d\{Bw,BBw) = d\{Au,BBw) < (f>(\max\ Id\{Su,TBw),-[d\{Au,Su)+d\{BBw,TBw)l$$

$$hd\{Au,TBw) + d\{BBw,Su)]\backslash)$$

$$= <l)\{d\{Bw,BBw)) < d\{Bw,BBw)$$

a contradiction. Hence *Bw — BBw = TBw* which shows that *Bw* is a common fixed point of the pair (S, *T).* Therefore, *Au* (= *Bw)* is a common fixed point of the pair *(B,T).* Similarly, one can show that *Su,Bw* and *Tw* are common fixed points of the mappings *A, S, B* and *T.* Uniqueness of the common fixed point follows easily. This completes the proof.

By choosing *A, B, S* and *T* suitably, one can deduce corollaries for a pair as well as for a triod of mappings. The detail of possible corollaries for a pair of mappings is not included because we have stated it as Theorem 2.2.1. We now outline the following natural results for pairs of three self mappings.

COROLLARY 2.3.1. Let yl,5 and r be self mappings of a symmetric (semi-metric) space *{X,d)* that enjoy *(b^)* (the Hausdorffness of r(d)). Suppose that

(6i8) *A(X)cT{X)US{X),*

(619) the pair *{A,T)* (or alternatively the pair *{A,S))* enjoys the property *(E.A),*

(620) the following inequality holds:

$$d\{Ax,Ay)<<l>\{m\{x,y)) \tag{2.5}$$

m{x, y) = max *U(Sx,Ty), ^[d{Ax, Sx) + d{Ay,Ty)], -[d{Ax, Ty) + d{Ay, Sx)]* I,

where 1 < fc < 2,

(621) *S{X)* (or alternately *T(X))* is a rf-closed (r(rf)-closed) subset *oi X.*

Then the pairs *{A, S)* and *{A, T)* have points of coincidence.

COROLLARY 2.3.2. Let *A,B* and *S* be self mappings of a symmetric (semi-metric) space *{X,d)* that enjoy (65) (the Hausdorffness of T(rf)). Suppose that

(M *A{X)cSiX),BiX)cSiX),*

(623) the pair *{B,S)* (or alternatively the pair *{A,S))* enjoys the property (£^.^4),

(624) the following inequality holds:

$$d\{Ax,By)<4,\{m\{x,y)) \tag{2.6}$$

m(x,*y)* = max *ld(Sx, Sy), -[diAx,Sx) + d(By,Sy)], ^[d{Ax,Sy) + d{By, Sx)\ * ,

where 1 < fc < 2,

(625) *S{X)* is a rf-closed (r(d)-closed) subset of *X.*

Then the pairs *[A, S)* and (B, *S)* have points of coincidence.

REMARK 2.3.1. A remark similar to Remark 2.2.2 in context of Theorems 2.3.1 and 2.3.2 can be furnished. But due to repetition, details are omitted.

We now give an example to illustrate the above theorem.

EXAMPLE 2.3.1. Consider *X* — [0,1] equipped with the symmetric d(x,y) *{x -yf.* Define

Ax^Bx = - ^ , if 0 < a; < 1, 5x = Tx = x, if 0 < re < 1,
1 + x

and (/»: 3?+ -> 3?+ as

(7^, i f O < « < l
m={
{I \it>i.

Then *A{X)* = *B{X)* = [0, i] C [0,1] = *S{X)* = *T{X),* *(f)* is nondecreasing and 0 < *(f){t)* < *t* for all *t e* (0, oo). Since *d* induces the usual topology therefore condition (be) is satisfied. The pair *{A, S)* satisfies the property *(E.A)* as there is a sequence {^} C [0,1] such that

lim *A* (-^ = lim ^ — = lim 5 (i) = lim - = 0 G *X.*
n->oo *\nj* n-»oo n + 1 *n-yoo \Tl J n-^oo n*

Also *{A, S)* is weakly compatible *as SO* = *AO* =^ *ASO* = *SAO.* *S{X)* = [0,1] is a rf-closed subset of *X.*

In order to verify contractive condition (2.4), if x = 0 and 0 < y < 1, then

diAx,Ay) = *(- ^)* < -^ - *(t>{y')* = *(f>{d{Sx,Sy))* < *<f>{m{x,y)).*
\i + *yj* i + r

In case *x* ^*y* and 0 < x < y < 1, then

d(Ax, Ay) = *y* 2 x-yp
1 + x 1 + 2/ ((l + x)(l+y))2

< / *\x-y* \ 2 ^ *\x-y\'*
yl + *\x-y\J* " l + |x-y|2

= *<j>{\x* - yH = *(f>{d{Sx,Sy))* < *<j){m{x,y)).*

Thus all the conditions of Theorems 2.3.1 and 2.3.2 are satisfied and 0 is the coincidence as well as common fixed point of the pair (A, *S).*

Finally, one may note that Theorem 2.3 due to Pant and Pant [109] cannot be used in the context of this example due to nonmetric setting besides other improvements realized due to certain tight conditions.

Chapter 3

ON COMMON FIXED POINTS OF HYBRID MAPPINGS UNDER STRICT CONTRACTIONS

3.1. INTRODUCTION

After the appearance of remarkable work due to Jungck [70], a multitude of common fixed point theorems were proved in several ways which include generalizations via improving contractive or contraction condition and enlarging the class of commutative mappings using suitable weak conditions of commutativity besides Ughtening the condition of continuity using 'reciprocal continuity' (cf. [105]) or by some other ways (e.g. [144]). These ways of proving new results continue to attract the attention of many researchers of this domain. With a view to enlarge the class of involved spaces, some researchers used semi-metric spaces to prove their results (cf. [47,67,74,98,130]). In this chapter, we also prove our results for pairs of hybrid mappings in semi-metric (potent semi-metric) spaces whose details are to be presented in Section 3.2.

It is well known that contractive conditions do not ensure the existence of fixed points unless the underlying space is assmned to be compact or the contractive conditions are replaced by relatively stronger conditions. Recently, Pant [102] initiated the study of common fixed points of noncompatible mappings under strict contractions whereas Aamri and Moutawakil [1] defined property *[E.A)* for self mappings which can also be viewed as somewhat unification of compatible and noncompatible mappings. Here it may be pointed out that a similar notion namely 'tangential maps' has already been introduced by Sastry and Mmthy [133] but this paper appears to have escaped the notice of the researchers of this domain. Let us recall that in Chapter 2 we have extended the results due to Aamri and Moutawakil [1] and Pant and Pant [109] to semi-metric (potent semi-metric) spaces. On the other hand, in 2004, Kanuran [82] extended the property *{E.A)* to hybrid pair of mappings and proved some coincidence and common fixed point theorems. More recently, Liu et al. [94] introduced the notion of common property *{E.A)* for single-valued and multi-valued mappings in metric spaces which contains property *(E.A)* as a special case to a pair of mappings as opposed to two pairs of mappings. Singh and Hashim [143] adopted property *(E.A)* to such hybrid pair of mappings which are defined on an arbitrary nonempty set with values in a metric space and used the same to prove their results.

The main purpose of this chapter is to extend common property *(E.A)* for pairs of hybrid mappings which are defined on an arbitrary nonempty set with values in a metric space and use the same to prove corrected and generalized versions of results due to Singh and Hashim [143] in semi-metric spaces. In process results due to Kasahara and Rhoades [86], Kamran [82] and Liu et al. [94] are extended to semi-metric spaces whereas Theorems 2.2.1-3 and 2.3.1-2 are generalized to hybrid pairs of mappings. Some related results are also derived besides furnishing an illustrative example which establishes the utility of our results over earlier ones.

The contents of the present chapter is as follows. Section 3.1 is introductory in nature in which we discuss the existing literature on coincidence and fixed points of hybrid pairs of mappings in metric as well as semi-metric spaces. In Section 3.2, we record the hybrid versions of some definitions-of weak commutativity in common fixed point theorems consideration. In the last section, we prove some results on coincidence and fixed points in semi-metric (potent semi-metric) spaces under strict contractions which improve and generalize several existing results besides furnishing an illustrative example.

3.2. SOME RELEVANT DEFINITIONS

In 1928, K. Menger defined semi-metric spaces as follows:

DEFINITION 3.2.1. A semi-metric on a set X is a function *d: X x X -^ [0,oo)* such that for all *x,y e X*

(ci) *d{x,y) = 0 4^x^y,*

(ca) *d{x,y) = d{y,x).*

The space *(X, d)* in which limiting points are defined in the usual way is called an *E-space.* The idea of £^-spaces goes back to Prechet. The pioneer work in this direction is essentially due to Wilson [154]. A topological space *{X,* r) is said to be symmetrizable iff there is a distance function *d* such that for any *AC. X,* *A = {x G X : d{x, A)* = 0}. In this case *d* is said to be semi-metric.

E. Cartan (1926) reserved the term symmetric space to those connected Riemannian manifold *M* where for each point *p £ M* there is an isometry of *M* that leaves *p* fixed and the differential of this isometry at *p* is multiplication by —1. Knowingly or unknowingly some authors (cf. [47,67]) call semi-metric spaces as symmetric spaces whereas others (cf. [98,130]) followed this terminology. In order to avoid any confusion, let us agree to rename the term referred as semi-metric thus far as potent semi-metric which is explicitly mentioned as follows:

If cf is a semi-metric on a set *X,* then for a; e X and e > 0, we write *B{x, e)* = *{y e X : d{x, y)* < e}. A topology *T{d)* on *X* is given by f/ 6 r(d) if and only if for each X e U, B(x, *e) C U* for some e > 0. A set 5 C X is a neighbourho6d *oi x e X* iff there exists *U* € *T{d)* such that *x e U C S.*

DEFINITION 3.2.2. A semi-metric *d* is said to be a potent semi-metric if for each *X E X* and for each e > 0, *B{x,* e) is a neighbourhood of *x* in the topology *T(d).* Thus a potent semi-metric space X is a topological space whose topology *T{d)* on *X* is induced by potent semi-metric *d.*

The distinction between a semi-metric and a potent semi-metric is evident as one can easily construct a semi-metric *d* such that *B{x, e)* need not be a neighbourhood of *X* in r(d). For a discussion on distinction between a semi-metric and symmetric space, one can see Arhangel'skii [4] and Burke [17] and references cited therein. Prom now on semi-metric space as well as potent semi-metric space will be denoted *hy{X,d).*

Let *{X, d)* be a semi-metric (potent semi-metric) space. Then following Nadler [101], we can adopt

(cs) *CL{X)* = *{A: A\s* nonempty closed subset of *X},*

(C4) *CB{X)* = *{A: A* is nonempty closed and bounded subset of *X},* *(c^)* For nonempty subsets *A, B* in *CB{X)* and *x ^ X,*

d{x,A) = ini{d{x,a) : a E A} and

H{A, B) = max{{sup *d{a, B) : a e A},* (sup *d{A, b) -.be B}}.*

It is also known that *{CB{X), H)* is a semi-metric space provided *{X, d)* is a semi-metric space (cf. [98]). Note that *H{Axn,A)* -)> 0 iff Ax„ -)• A G *CL{X)* in the topology *T{H).*

As mentioned earlier, the notion of weak commutativity in metric fixed point theory was introduced by Sessa [136]. After this definition, researchers of this domain also introduced several such weak conditions namely: Compatible mappings. Compatible mappings of type (A), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), Biased maps, R-weakly commuting mappings and some others whose systematic comparison and illustration can be found in Murthy [100]. The following definitions of weak commutativity are relevant in the present context and can be naturally adopted to the setting of semi-metric (potent semi-metric) spaces.

DEFINITION 3.2.3. [85] Let *{X,d)* be a semi-metric (potent semi-metric) space, *F : X ^ X* and *T : X ^ CL{X).* A pair (F, *T)* of hybrid mappings is said to be compatible if *FTx e CL(X)* for all a: G AT and

$$\lim_{n \to oo} H\{FTxn,TFxn) = 0$$

whenever *{xn}* is a sequence in *X* such that $\lim_{n \to oo}$ *Fx^ — t^A—* $\lim_{n—KX>}$ *Txn-*

Here it may be noted that pointwise /?-weakly commuting mappings (cf.[102]) need not be compatible. Also, on the points of coincidence i?-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points of contractive type mappings.

DEFINITION 3.2.4. [51] Let *{X,d)* be a semi-metric (potent semi-metric) space, *F : X ^ X* and T : X —^ *CL{X).* A pair of mappings (F, T) is said be quasi-coincidentally commuting if for all coincidence points *"x"* of *(F,T), FTx* C *TFx.*

DEFINITION 3.2.5. [51] Let *{X,d)* be a semi-metric (potent semi-metric) space, *F : X* ^ *X* and *T : X* ^ *CL{X).* A mapping *F* is said to be coincidentally idempotent w.r.t mapping T, if *F* is idempotent at the coincidence points of the pair(i^,T).

Recently, Lin et al. [94] defined the common property *{E.A)* for two pairs of single-valued mappings and hybrid mappings in metric spaces. Now, we extend common property *{E.A)* for pairs of single-valued and multi-valued mappings on an arbitrary nonempty set *Y* with values in metric space *{X,d).*

DEFINITION 3.2.6. Let *{X,d)* be a metric space, *Y* be an arbitrary nonempty set and *F,G,S,T .Y* ^ *X.* Then the pairs (F, *G)* and (5, T) of mappings are said to have common property *{E.A)* if there exist two sequences {x„}, {?/„} in *Y* and some t G X such that

$$\lim_{n \text{—\^{}oc}} Fxn = \lim_{n \text{—>oo}} Gxn = \lim_{n \text{—\^{}oo}} Sxjn = \lim_{n \text{—¥oo}} Tyn = t.$$

DEFINITION 3.2.7. Let *{X,d)* be a metric space, *Y* be an arbitrary nonempty set, *F,G : Y* ^ *X and S,T : Y* -^ *CL{X).* Then the pairs (F, 5) and *{G,T)* of hybrid mappings are said to have common property *{E.A)* if there exist two sequences {x„}, {?/„} in *Y,* some *t* ^ *X* and *A,B*^ *CL{X)* such that

$$\lim_{n \text{—>oo}} Sxn \text{ — } A, \quad \text{hm}_{n \text{—>oo}} \text{ Ty„} = \text{B}, \quad \lim_{n \text{—>oc}} Fxn = \lim_{n \text{—¥oc}} Gi/n = t \wedge ACi\ B.$$

Clearly, compatible as well as noncompatible pairs of mappings satisfy property *{E.A).*

EXAMPLE **3.2.1.** Let *X* = [1,-1-00) be a metric space equipped with the usual metric. Define *F,G : X* ^ *X and S,T : X* ^ *CB{X)* by *F{x)* = 2 + a;/3, *G{x)* = *2 + x/2, S{x)* = [1,2 + *x]* and *T{x)* = [3,3 + *x/2* for aU *x e X.* Consider the sequences {x„} = {3 + *1/n}, {yn}* = *{2-\- l/n}.* Clearly,

$$\lim Sxn = [1,5] = A, \quad \text{hm } Tyn = [3,4] = B, \quad \lim \text{Fx„} = \lim Gy\wedge = \wedge e\ A\ D\ B.$$

Therefore, the pairs *{F,S)* and *{G,T)* are said to satisfy the common property *(E.A).*

3.3. RESULTS FOR HYBRID PAIRS OF MAPPINGS

Recently, Singh and Hashim [143] proved some results for pairs of hybrid mappings which contain some errors. Indeed, results in [143] are true up to coincidence points but claim of existence of common fixed points cannot be established for an arbitrary set *Y* due to the fact that *Sz* ^ *Y* and hence one cannot hypothesize *SSz* = 52. Similar error also occurs in the context of (/r)-commuting as well (cf. [65]). For the sake of completeness, we state the following theorem due to Singh and Hashim [143] proved for a pair of hybrid mappings defined on an arbitrary nonempty set *Y* with values in a metric space *{X,d).* Let us denote set of coincidence points ofthepair (A5)byC(yl,5).

THEOREM 3.3.1. Let .4 : r -> *CL{X)* and 5 : y -> X be such that

(ce) *A{Y)CS{Y*

(cy) the pair *{A,S)* satisfies the property (£^.^4),

(cg) *H{Ax, Ay)* < *Tn{x, y)* when *m{x, y)* > 0, where

m{x, y) = max{d(5'x, *Sy), a[d{Sx, Ax)+d{Sy, Ay)],a[d{Sy, Ax)+d{Sx, Ay)]},*

0 < a < 1.

If *A{Y)* or *S{Y)* is a complete subspace of *X,* then *C{A, S)* is nonempty.

Further, *A* and *S* have a common fixed point provided that *SSz* = *Sz* and *A* and *S* are (/T)-commuting at z G *C{A, S).*

We state and prove our results as follows:

THEOREM 3.3.2. Let *{X,d)* be a semi-metric (potent semi-metric) spa^e with condition *(be)* (Hausdorffness of r(d)). *Let F,G : Y -^ X and S,T : Y -^ CL{X)* such that

(cg) the pairs (F, 5) and *{G,T)* have the common property *{E.A),* (cio) for all X 7«^ y e y,

HiSx, Ty) < max id{Fx, Gy), ^[d{Fx, Sx) + d{Gy,Ty)], ^[d(Gy, 5a:) + d{Fx,Ty)]\ ,

(3.1)

1 < it < 2.

If F{Y) and G{Y) are d-closed (r(d)-closed) subsets of X, then the pairs {F, S) and (G, T) have points of coincidence.

Moreover, ii Y — X and the pairs {F, S) and {G, T) are quasi-coincidentally commuting and coincidentally idempotent, then F,G,S and T have a common fixed point in X.

PROOF. Firstly, one needs to note that a sequence {xn} in a potent semi-metric space {X,d) converges to a point x in T{d) iff d{xn,x) —>• 0. To substantiate this, suppose Xn -> X and let c > 0. Since B{x,e) is a neighbourhood of x there exists U G r(d) such that x e U G B{x, e). Since x„ ^ x there is a m G A^ (the natural number) such that x„ G f/ C B{x,e) for n > m so rf(x„,x) < e for n > m i.e. d{xn, x) -> 0. The converse part is obvious in view of the definition of T{d).

In view of (cg), there exist two sequences {x„}, {y„} in Y and t E X, A,Be CL{X) such that

hm Sxn = A, Um ry„ = B, lim Fx„ = lim Gyn = te Ar\B.
n->oo

As F{Y) and G(y) are d-closed subsets of X and Fx„ -^ i, Gy„ ->• <, then one can find points u,v EY such that Fu = t = Gv. Now we show that Fu € 5u and Gu G Tu. If not, then using inequality (3.1), one obtains

H{Sxn,Tv)

< max |d(Fx„, Gu), ^KFx„, 5x„) + d{Gv,Tv), ^[d(Gv, 5x„) + rf(Fx„,rt;)]|.

On letting n —>• oo, we have

H(A,Tv) < ^d{Gv,Tv) < d{Gv,Tv)

or d{Gv,Tv) < H{A,Tv) < d(Gv,Tv),

a contradiction. Hence *Gv* € *Tv* which shows that the pair (G, T) has a point of coincidence.

For *Fu e Su,* again using inequality (3.1), one gets

d{Fu, Su) = *d{Gv, Su)* < *H(Tv, Su)*

< max *<d{Fu,Gv), ^[d{Fu,Su)* + *d{Gv,Tv), ^[d{Gv,Su)* + *d{Fu,Tv)]*

= *k* *-d{Fu, Su)* < *d{Fu, Su),*

a contradiction. Hence *Fu* e *Su* which shows that the pair (F, *S)* has a point of coincidence.

Since u is a point of coincidence of the pair (G, *T),* therefore using quasi-coincidentally commuting property of *{G,T)* and coincidentally idempotent property of *G* w.r.t. *T*, one can have

GVGTV and *GGv* = *Gv,*

therefore *Gv* = *GGv e G(Tv)* C *T{Gv)* which shows that *Gv* is the common fixed point of mappings *G* and *T.*

Since the pair *(F, S)* is quasi-coincidentally commuting, coincidentally idempotent and *u* is the coincidence point of the pair (F, *S),* i.e.

Fu e *Su* and *FFu* = *Fu,*

therefore *Fu* = *FFu E F{Su)* C *S{Fu)* which shows that Fit is the common fixed point of the pair (F, 5). But *Fu* = *Gv* — *t,* hence f is a common fixed point of mappings F, *G, S* and *T.* The same proof works for alternate statement as well. This completes the proof.

REMARK 3.3.1. Theorem 3.3.2 is a corrected and improved version of Theorem 3.3.1 due to Singh and Hashim [143] and also extends results of Liu et al. [94] and Kamran [82] to semi-metric spaces beside generalizing results of Chapter 2 to two pairs of hybrid mappings. Theorem 3.3.2 also generalizes some results due to Aamri and Moutawakil [1], Pant and Pant [109], Kasahara and Rhoades [86] and others to two pairs of hybrid mappings in metric spaces without requiring suitable containment between range sets of involved mappings under more general contractive condition.

For a pair of hybrid mappings, we have the following.

THEOREM 3.3.3. Let *{X,d)* be a semi-metric (potent semi-metric) space with condition (be) (Hausdorffness of r(d)). *Let F :Y ^ X and T :Y -^ CL{X)* such that

(cii) the pair (F,T) has the property *{E.A),*

(C12) for *a\\x ^y eY*

HiTx, Ty) < max *UFX,Fy), '^[d{Fx,Tx) + d{Fy,Ty)],* ^[d(Fy,*Tx) + d{Fx, Ty)]* , (3.2)

1 < fc < 2.

If *F{Y)* is a d-closed (T(d)-closed) subset of *X*, then the pair *{F, T)* has a point of coincidence.

Moreover, *MY* = *X* and the pair *{F,T)* is quasi-coincidentally commuting and coincidentally idempotent, then *F* and *T* have a common fixed point.

The following form of Theorem 3.3.3 also remains true.

THEOREM 3.3.4. Theorem 3.3.3 remains true if d-closedness (r(rf)-closedness) of *F{Y)* is replaced by d-closedness (T(d)-closedness) of T(y) along with *T{Y)* C *F{Y)* retaining the rest of the hypotheses.

Since a pair of noncompatible mappings satisfies property *{E.A),* therefore we have the following.

COROLLARY 3.3.1. Let (X, *d)* be a semi-metric (potent semi-metric) space with condition (feg) (Hausdorffness of *r{d)).* Let F : r -> X and T : F ->• *CL{X)* such that

(cjs) the pair (F, *T)* is noncompatible,

(CH) the pair (F, T) satisfies inequality (3.2).

If *F{Y)* is a rf-closed (r(d)-closed) subset of *X*, then the pair *{F, T)* has a point of coincidence.

Moreover, *liY* = *X* and the pair (F,*T)* is quasi-coincidentally commuting and coincidentally idempotent, then *F* and *T* have a common fixed point.

REMARK 3.3.2. Theorem 3.3.3 presents a substantially improved version of results due to Kamran [82, Theorems 3.4 and 3.10] and also Theorem 3.3.1 due to Singh and Hashim [143].

REMARK 3.3.3. if we set r to be a single-valued mapping in Theorem 3.3.3, then we get an improved version of Theorem 2.2.1 which never requires containment of range of one map into the range of other.

By choosing mappings F, G, *S* and *T* suitably in Theorem 3.3.2, one can obtain results for a pair as well as for two pairs involving only three mappings. We have already stated results for a pair of mappings. We now deduce two corollaries for two pairs formed from three mappings.

COROLLARY 3.3.2. Let (X, *d)* be a semi-metric (potent semi-metric) space with condition (fee) (Hausdorffness of r(d)). *Let F :¥ ^ X a.nd S,T :¥ -^ CL{X)* such that

(cis) the pairs (F,*S)* and *{F,T)* share the common property *{E.A),*

(cie) for all a; 7^ y G y,

H{Sx, Ty) < max *L{Fx, Fy), ^[d{Fx, Sx)* + *d{Fy, Ty)], '^[diFy, Sx)* + *d{Fx, Ty)]* , (3.3)

1 < A; < 2.

If *F{Y)* is a d-closed (r(rf)-closed) subset of X, then the pairs (F, *S)* and (F, *T)* have points of coincidence.

Moreover, *ii Y* = *X* and the pairs (F, *S)* and (F, *T)* are quasi-coincidentally commuting and coincidentally idempotent, then F, *S* and *T* have a common fixed point in X.

COROLLARY 3.3.3. Let *{X,d)* be a semi-metric (potent semi-metric) space with condition *(h)* (Hausdorffness of T(rf)). *Let F,G : Y -^ X a.nd S : Y -^ CL{X)* such that

(cir) the pairs (F, 5) and (G, 5) share the common property *{E.A)^*

(cis) for all X 7^ y G Y,

H{Sx, Sy) < max |d(Fx, *Gy), \\d{Fx, Sx)* + rf(Gy, 5i/)], *\[d{Gy, Sx)* + d(Fx, 5y)]|,

(3.4)

1 < A; < 2.

If *Fiy)* and *G{y)* are d-closed (r(d)-closed) subsets of *X*, then the pairs (F, 5) and (G, 5) have points of coincidence.

Moreover, *li Y = X* and the pairs (F, 5) and (G, *S)* are quasi-coincidentally commuting and coincidentally idempotent, then F, G and 5 have a common fixed point in *X*.

We now furnish an example to demonstrate the validity of the hypotheses and degree of generality of our results over earlier ones especially those contained in [1,82,86,143]. Our example presents a nonmetric setting satisfying the hypotheses of Theorem 3.3.3 which in turn establishes the genuineness of our results over all relevant metrical common fixed point theorems.

EXAMPLE 3.3.1. Consider *X* = [0,1] equipped with the semi-metric *d{x,y)* = *{x - yf*. Define *F : X ^ X axidT : X ^ CL{X)* as follows :

FO = 0, Fx = 1 for *X* G (0,1] and *Tx* = *[1 - x,* 1], for all *xeX.*

Clearly *F{X)* = {0,1} is rf-closed in *X*. The pair (F, T) has property *(E.A)* as for the sequence {1 — ^} c[0,l], we have

lim F (1 - i) = 1 e [0,1] = lim *T (I - -]* .
n-4oo \^ *nj* *n->oo* \ *u)*

By a routine calculation one can verify the contractive condition (3.2) for every *X ^y ^ X*. Also notice that the 1 = Fl G Tl. Since the topology induced by *d* is usual on [0,1], it will be Hausdorff and therefore condition *(be)* is naturally satisfied. Thus all the conditions of Theorem 3.3.3 are satisfied and 1 is the common fixed point of *F* and *T*. Here, one needs to note that *d* is not a metric as *d{0,*1) = 1 > ^ + I = d(0, |) + d(|, 1). Thus all the available metrical common fixed point theorems cannot be used in the context of this example.

The following theorem involves a control function *(j) : R*^ —^ ^^ which satisfies the properties (612) and (613) in Chapter 2.

THEOREM 3.3.5. Let *{X,d)* be a semi-metric (potent semi-metric) space with condition *(be)* (Hausdorffness of T(d)). *Let F,G : Y ^ X and S,T : Y ^ CL{X)* such that

(cig) the pair *{F,S)* (or *{G,T))* has the property *{E.A),*

(c2o) *S{Y)* c *GiY)* (or T(r) c *F{Y)),*

(c2i) for *allx^y EY,*

$$H\{Sx,Ty) < cj>\{m\{x,y)), \tag{3.5}$$

m{x, y) = max *[d{Fx, Gy), ^[d{Fx, Sx)* + *d{Gy, Ty)],* ^[d(Gy, *Sx)* + *d{Fx, Ty)]* ,

1 < A; < 2.

If *F{Y)* and *G{Y)* are d-closed (r(<i)-closed) subsets of *X,* then the pairs (F, *S)* and *{G, T)* have points of coincidence.

Moreover, *iiY* = *X* and the pairs (F, *S)* and (G, *T)* are quasi-coincidentally commuting and coincidentally idempotent, then *F,G,S* and *T* have a common fixed point in *X.*

PROOF. Suppose the pair *{G,T)* has property *(E.A),* then there exists a sequence *{xn}* in *Y* such that

$$\underset{n\to\infty}{\mathrm{hm}}\ Gxn\ {}^\wedge t\ e\ A^\wedge\ \lim_{n\to\infty} Txn, \quad \text{for some } t\ e\ X \text{ and } A\ G\ CL\{X).$$

Since *T{Y)* C *F{Y),* for each {x„} there exists {y„} such that Fy„ G rx„ and hm *Fyn* = *t e A* = hm Ta:„. Thus, we have Fy^ -^ t, Gx„ -> f and *Txn* -^ *A.*

Now, we assert that 5y„ -^ *A*. If not, there must exists a subsequence *{Sym}* of *{Syn},* a +ve integer *N* and a real number 7 > 0 such that for each *m > N, we* have *H{Sym, A)* > 7. Using inequality (3.5), we get

$$H\{Sym,\ Txm) < 0(\max \{rf(Fy\wedge,\ Gxm),\ \overset{k.}{-}\wedge[diFym,\ Sym) + d\{Gxm,\ Txm)i$$

$$\frac{k}{2}[d\{Gxm,Sym) + diFym\wedge Txm)]\})$$

which on letting *m -> 00,* reduces to

$$H\{Sym,A)<<f>(\wedge d\{t,Sym)$$

$$<(f>(\wedge H\{A,sym)]<H\{A,Sym),$$

a contradiction. Hence *Syn -> A.*

Thus the pairs (F, 5) and (C?,T) share common property *(E.A).* The rest of the proof of this theorem can be completed on the Unes of Theorem 3.3.2. This concludes the proof.

COROLLARY 3.3.4. Let *{X, d)* be a semi-metric (potent semi-metric) space with condition *%)* (Hausdorffness of T(d)). Let F,G : r -» X and 5,T : F -> *CL{X)* such that

(C22) the pairs (F, 5) and (G,T) have the common property *{E.A),*

(C23) for all X 7^ y e F,

$$H\{Sx,Ty) \quad < \quad (f>\{\max\{d\{Fx,Gy),d\{Fx,Sx),diGy,Ty),d\{Gy,Sx),d\{Fx,Ty)\}).$$

If *F{Y)* and *G{Y)* are d-closed (T(d)-closed) subsets of X, then the pairs (F, *S)* and *{G,T)* have points of coincidence.

Moreover, *ii Y ^ X* and the pairs (F, *S)* and (G, *T)* are quasi-coincidentally commuting and coincidentally idempotent, then F, *G, S* and *T* have a common fixed point in *X.*

PROOF. The proof follows from Theorem 3.3.5 in view of the following observation when *k = 2.*

$$H\{Sx,\ Ty) < ((>(\max\{d\{Fx,\ Gy),\ d\{Fx,\ Sx),\ d\{Gy,\ Ty),\ d\{Gy,\ Sx),\ d\{Fx,\ Ty)\})$$

$< \ <f>imax\{d\{Fx,GyU(Fx, Sx) + d\{Gy,Ty),d\{Gy, Sx) + d(Fx, Ty)\})$.

REMARK 3.3.4. Corollaxy 3.3.4 extends the relevant result due to Liu et al. [94] to semi-metric (potent semi-metric) spaces and results due to Zhu et al. [157] for hybrid pairs of mappings.

Finally, we state the following result for two pairs of single-valued mappings satisfy the common property *(E.A).*

THEOREM 3.3.6. Let *{X,d)* be a semi-metric (potent semi-metric) space with condition *(be)* (Hausdorffness of r(rf)). Let *F,G,S,T :Y -^ X he* four single-valued mappings satisfy condition (cg) and

(C24) for *a\\ X :^y eY,*

$d\{Sx,Ty) < \max |d(Fx,Gy), \hat{}[d(Fx,Sx) + d\{Gy,Ty)], |[d(Gy,Sx) + d\{Fx,Ty)]\backslash$,

$1 < fe < 2.$

If *F{Y)* and *G{Y)* are d-closed (r(rf)-closed) subsets of *X,* then the pairs (F, *S)* and *{G, T)* have points of coincidence.

Moreover, *iiY = X* and the pairs (F, *S)* and *{G, T)* are weakly compatible, then F, G, *S* and *T* have a common fixed point in *X.*

REMARK 3.3.5. Results similar to Corollaries 3.3.2 and 3.3.3 can be outhned in respect of Theorem 3.3.5 and Corollary 3.4.4 for two pairs of mappings formed from three mappings.

REMARK 3.3.6. All the results of this chapter remain true, if one replaces

$\max \backslash d\{Fx, Gy), \hat{}[d\{Fx,Sx) + d\{Gy,Ty)], \hat{}[d\{Gy,Sx) + d\{Fx,Ty)]\backslash , \quad 1 < fc < 2$

by

$\max \{d\{Fx, Gy), k[d\{Fx, Sx) + d\{Gy, Ty)], k[d\{Gy, Sx) + d\{Fx, Ty)]\}, \quad 0 < k < I.$

Chapter 4

IMPACT OF PROPERTY (E.A) AND COMMON PROPERTY (E.A) ON COMMON FIXED POINT THEOREMS

4.1. INTRODUCTION

Motivated by the fact that a fixed point of any mapping on metric spaces can always be viewed as a common fixed point of that mapping and identity mapping on the same space. Jimgck [70] proved the following interesting generalization of celebrated Banach contraction principle. While proving his result, Jungck [70] replaced identity mapping with a continuous mapping.

THEOREM 4.1.1. Let r be a continuous mapping of a complete metric space *{X,d)* into itself. Then T has a fixed point in *X* if there exists *k* 6 (0,1) and a mapping *S : X* -^ *X* which commutes with *T* and satisfies *S{X)* C *T{X)* and *d{Sx,Sy)* < *kd{Tx,Ty),* for all *x,yeX.*

As reflected in Theorem 4.1.1, a metrical common fixed point theorem generally involves conditions on commutativity, continuity, completeness and suitable containment of ranges of the involved mappings besides an appropriate contraction condition and researchers in this domain are aimed at weakening one or more of these conditions.

As discussed in earlier chapters, after the evolution of weak commutativity of Sessa [136] and compatibiUty of Jungck [72], researchers started utilizing weak conditions of commutativity as a tool to improve common fixed point theorems. Consequently, the recent Uterature of metric fixed point theory has witnessed the evolution of several weak conditions of commutativity whose lucid survey and illustration (upto 2001) is available in Murthy [100]. In what follows, we choose to utilize the most natural of these weak conditions namely weak compatibility due to Jungck [74].

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With a view to improve the continuity requirement in fixed point theorems, Kannan [84] proved a result for self mappings (without continuity) and shown that there do exist mappings which are discontinuotis at theii fixed points. However, common fixed point theorems without any continuity requirement were established by Singh and Mishra [144] (also Pant [105]). Here, we opt a method which is essentially inspired by Singh and Mishra [144] wherem the completeness of the space is alternately replaced by the completeness of range of one or more mappings as required.

The tradition of improving contraction conditions in fixed and common fixed point theorems is still in fashion and continues to be most effective tool to improve such results. For an extensive collection of contraction conditions one can be referred to Rhoades [126,128] and references cited therein. Most recently, with a view to accommodate many contraction conditions, Popa [122] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. In this chapter, we also utihze implicit functions to prove our results because of their versatihty of deducing several contraction conditions at the same time. This fact will be substantiated by furnishing several examples in Sections 4.2 and 4.5.

Most recently, Aamri and Moutawakil [1] and Liu et al. [94] introduced the notions of property *(E.A)* and common property *(E.A)*, respectively. As mentioned earlier, a pair of compatible as well as noncompatible self mappings of a metric space *(X,d)* satisfies the property *(E.A)*. In general, pairs enjoying property *(E.A)* and common property *(E.A)* need not follow the pattern of containment of range of one mapping into the range of other as utilized in common fixed point considerations but still it relaxes such requirements.

EXAMPLE 4.1.1. Consider $X = [-1,1]$ with the usual metric. Define the self mappings T and S on X as follows :

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x = -1 \\ \% & \text{if } -1 < x < 1 \\ \frac{3}{5}, & \text{if } x = 1 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 1, & \text{if } x = -1 \\ \frac{1}{2}, & \text{if } -1 < x < 1 \\ \frac{-1}{2}, & \text{if } x = 1. \end{cases}$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly,

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0.$$

Then T and S satisfy property $(E.A)$. Also, $T(X) = \{1, \frac{3}{4}\} \cup (\frac{1}{2}, 1)$ and $S(X) = [\frac{1}{2}, 1]$. Here one needs to note that neither $T(X)$ is contained in $S(X)$ nor $S(X)$ is contained in $T(X)$.

In this chapter, we observe that up to a pair(s) of mappings newly introduced property $(E.A)$ (common property $(E.A)$) buys containment conditions on ranges without any continuity requirement besides limiting commutativity requirement to the points of coincidence. Moreover, the completeness requirement of the space is weakened to alternative natural conditions and the involved contraction condition is replaced by implicit functions. In Section 4.5, we also define a new implicit function to enhance the domain of applicability which includes several well known contraction conditions such as: Ciric quasi-contraction, generalized contraction, ϕ-type contraction, rational inequality and others besides admitting new unknown contraction conditions which is used to prove a general common fixed point theorem for two pairs of weakly compatible self mappings satisfying the common property $(E.A)$. In process, many known results are enriched and improved. Some related results are also derived besides furnishing illustrative examples.

The details of implicit function due to Popa [122] will be presented in Section 4.2. In Section 4.3, we prove our results using the implicit function (due to Popa [122]) where Section 4.4 is devoted to some illustrative examples to the results proved in Section 4.3. This chapter concludes with Section 4.7 wherein some examples illustrating the results presented in Section 4.6.

4.2. IMPLICIT FUNCTION I

Recently, Popa [122] introduced the idea of implicit functions to prove new common fixed point theorems. To describe the implicit functions of Popa [122], let Φ be the family of real lower semi-continuous functions $F(t_1, t_2, \cdots, t_6) : \mathbb{R}^6 \to \mathbb{R}$ satisfying the following conditions:

F_1 : F is nonincreasing in the variables t_5 and t_6, i*2 : there exists *hG* (0,1) such that for every *u,v >0* with

F2a '• *F{u,V,v,u,u + v,0) <0* or

•^26 : *F{u,V,u,v,0,u + v) <0*
we have *u < hv,* and

F3 : *F{u, u,* 0,0, It, *u)* > 0, for all u > 0.

The following examples of such functions appeared in Popa [122] with details and verifications.

EXAMPLE 4.2.1. Define F(fi,i2,--,^6) :3fit ^3?as

F{ti,t2,...,te) = *ti-kmaxlt2,t3,U,-{t5* + *te)[,* where A;G(0, 1).

EXAMPLE 4.2.2. *De&neF{ti,t2,...,k)* *-.^l^Uas*

F{ti,t2,... ,te) = *tl* - *ci* *max{tl, tj, tl}* - *C2*maxl^afs,^4^6} - *czt^te,*

where Cj > 0, C2, C3 > 0, Ci + 2c2 < 1 and ci + C3 < 1.

EXAMPLE 4.2.3. *De^neF{tut2,...,k):^+^^as*

where a > 0 , ft, c, d > 0 , *a* + *h* + *c<l* and a + rf < 1.

EXAMPLE 4.2.4. Define F(ii,i2,--,^6) :5R^ ^3?as

F(ii, i2,.. •, ie) =*t\-* *at\t2* - ^^1*3*4 - *ct%* - *dhtl,*

where a > 0 , 6,c,d>0, a + c + d < l and a + 6 < 1.

EXAMPLE 4.2.5. Define F(ii,i2,.--,<6) :9fJt -^3?as

F(i„i2,...,i6) = ^ ? - A : ^ - : ^ ^ | ^ , where A:e(0,l).

EXAMPLE 4.2.6. DefineF(^i,t2,...,i6) : 3?+->-3?as

F(ii,f2,...,<6) = t ? - a i 2 - --,——1^-72' where a > 0,6 > 0 and a + 6 < 1.

Implicit functions are quite fruitful in deducing many known contraction conditions besides admitting new ones. To substantiate this view point we add some examples described as follows:

EXAMPLE 4.2.7. *De&aeF{tut2,...,h):^l-^^as*

F{tut2,...,h) = *ti-kmaxlt2,tz,h,-^,-^>,* where A;G(0,1).

Fi : Obviously.

F2a : Let *u> 0* and *F{u,v,v,u,u + v,0)* = *u-kmax{VyV,u,^{u + v)}* < 0. If w > u, then *u* < *ku* < *u, a.* contradiction. Thus *u* <*v* and *u* < *kv,* where fee (0,1).

F2b •• Similar argument as in *F2a-*

Fa : *F{u,u,*0,*0,u,u)* = *u* - *ku* = *{1* - *k)u* > 0, for all *u> 0.*

EXAMPLE 4.2.8. Define *F(tu* «2, • • •, *te).*- 3?^ -^ 3? as

F{ti,t2,...,t6) = *ti-kmaxlt2,* ^ \ ^ ^ S, where fc G (0,1).

EXAMPLE 4.2.9. Define F(fi,t2,. •-.te) :3f?^-^•SR as

F{ti,t2,..., te) = *i - *{o-it2* + ^2*3 + *aiU* + *(^ih* + *o-bk),* where $\sum_{i=1}^{5}$ a j < 1.

EXAMPLE 4.2.10. Define F(ti,t2,...,t6) :3?^-4g?as

F{ti,t2,...,te) = *ti--*$\frac{k}{}$*max{t2,t3,U,t5,te},* where A; G (0,1).

Since verifications of requirements (Fi, F2 and F3) for Examples 4.2.8-4.2.10 are straightforward, hence details are omitted.

4.3. RESULTS VIA PROPERTY (E.A)

Here, we prove a general common fixed point theorem for a pair of self mappings using the implicit function due to Popa [122] without requiring the condition on containment of ranges of the involved mappings. We also utilize our main theorem to highlight how several fixed point theorems can be unified by using an implicit function.

THEOREM 4.3.1. Let 5 and T be self mappings of a metric space *{X,d)* such that

(di) the pair *{S^T)* satisfies the property *{E.A),*

{6.2) for all a;, 2/ 6 X and F G *,

$$F\{d\{Tx,Ty),d\{Sx,Sy),d\{Sx,Tx),d\{Sy,Ty),d\{Sx,Ty),d(Sy,Tx)) \quad < \quad 0, \tag{4.1}$$

(da) *S{X)* is a complete subspace of *X.*

Then the pair (5, T) has a point of coincidence. Moreover, the pair (5, T) has a common fixed point provided it is weakly compatible.

PROOF, in view of (di), there exists a sequence {xn} in *X* such that

$$\underset{n\text{-}yoo}{\mathrm{Um}}\ Txn = \underset{n\longrightarrow oo}{\lim}\ 5x_{,,} = t \pounds X.$$

As *S{X)* is a complete subspace of *X,* every convergent sequence of points of *S{X)* has a Umit in *S(X).* Therefore

$$\underset{n\longrightarrow oo}{\mathrm{hm}}\ Sxn — t = Sa = \underset{n\longrightarrow oo}{\lim}\ Tx_{,,}, \quad \text{for some a € X}$$

which in turns yields that $t = Sa\ e\ S\{X).$ Now assert that $Sa = Ta.$ If it is not, then $d\{Ta,Sa) > 0.$ Using (4.1)

$$F\{d\{Ta, Txn), diSa, 5z_{,,}), d\{Sa, Ta), d\{Sxn, Txn), d\{Sa, Tx^{\wedge}), d(5x_{,,}, Ta)) < 0$$

which on making n -4 oo, reduces to

$$F\{d\{Ta,t), d\{Sa, t), d\{Sa, Ta), d\{t,t), d\{Sa, t), d\{t, Ta)) < 0$$

$$\text{or} \quad F\{d\{Ta, Sa), 0, d(Sa, Ta), 0,0, d\{Sa, Ta)) < 0,$$

yielding thereby (due to *F2b),* $d\{Ta, Sa) < 0.$ Hence $Ta = Sa$ which shows that *a* is a coincidence point of *S* and *T.*

Since *S* and *T* are weakly compatible, then

$$St = STa = TSa = Tt.$$

Now assert that $Tt = t$. If not, then $d\{Tt,t) > 0$. Again using (4.1)

$$F\{d\{Tt,Ta),d\{St,Sa)J\{St,Ttld\{Sa,Ta),d(St,Ta),d\{Sa,Tt)) \quad < \quad 0$$

$$\text{or} \qquad F\{d\{Tt,t), d\{Tt,t), 0,0, d\{Tt, t), d\{t, Tt)) < 0,$$

which contradicts $F3$. Hence $Tt = t$ which shows that f is a common fixed point of S and T. The uniqueness of the common fixed point is an easy consequence of inequahty (4.1). This completes the proof.

REMARK 4.3.1. Theorem 4.3.1 is a generahzed and improved form of Theorem 4.1.1 due to Jungck [70] without any continuity requirement besides relaxing the containment of the range of one map into the range of the other (i.e $T\{X)$ C $S\{X))$. Also the commutativity requirement is reduced to points of coincidence along with replacement of the completeness of the space with alternate natural condition. In fact Theorem 4.3.1 refines all existing results estabhshed for a pair of mappings.

COROLLARY 4.3.1. The conclusions of Theorem 4.3.1 remain true if for all x,y G X inequahty (4.1) is replaced by any one of the following:

$$(\hat{}4) \qquad d\{Tx,Ty) \quad < \quad kmaxld\{Sx,Sy),d\{Sx,Tx),d\{Sy,Ty),\hat{}[d(Sx,Ty)+d\{Sy,Tx)]\backslash,$$

where k e $(0,1)$.

$$(\hat{}5) \quad d\hat{}\{Tx, Ty) < d \text{ maxK(5a;, } Sy), d'\hat{}\{Sx, Tx),d'\hat{}\{Sy, Ty)\}+C2\ max\{d\{Sx, Tx)$$

$$d\{Sx, Ty), d\{Sy, Ty)d\{Sy, Tx)\} + c\hat{}diSx, Ty)d\{Sy, Tx),$$

where ci > 0, C2,03 > 0, Ci + 2c2 < 1 and Ci + C3 < 1.

$$\{de) \qquad d\hat{}\{Tx,Ty) \quad < \quad d\{Tx,Ty)[ad\{Sx,Sy)+bd\{Sx,Tx)+cd\{Sy,Ty)]+d.d\{Sx,Ty)d\{Sy,Tx$$

where $a > 0$, $b,c,d>0$, a + 6 + c < l and $a + d <l$.

$$(dr) \quad d\backslash Tx, Ty) < ad'\hat{}\{Tx, Ty)d\{Sx, Sy)+hd\{Tx, Ty)d\{Sx, Tx)d\{Sy, Ty)$$

$$+ cd\hat{}\{Sx, Ty)d(Sy, Tx) + d.d\{Sx, Ty)d\hat{}\{Sy, Tx),$$

where a > 0, 6, c, d > 0, a + 6 < 1 and $a + c + d<l$.

$$\begin{matrix}(H\ \ ) & H\hat{}(T.\ T,A & <, & ,.\dfrac{\hat{}(g\hat{}.Tx)\hat{}\{Sy,Ty) + cfjSx,Ty)<f\{Sy,Tx)}{,\hat{}a\hat{}Sx,Sy) + diSx,Tx) + diSy,Ty)} & . & . & .\hat{}\ \hat{}. \\ ids) & d\ (Tx,Ty) & < & k & ' & ''\hat{}'''\hat{}\ \hat{} & (°'\ ')\bullet\end{matrix}$$

where a > 0, 6 > 0 and *a + b < 1.*

(dio) d(rx,Tj/) < fcmax *Usx, Sy),d{Sx,Tx),d{Sy,Ty), ^d{Sx,Ty), ^d{Sy,Tx)* ,

where A; G (0,1).

(dn) <Tx,Tt/) < *kmaK[d{Sx,Sy),^[d{Sx,Tx) + d{Sy,Ty)],^[diSx,Ty)*

+d{Sy,Tx)], where A; G (0,1).

(dia) *d{Tx,Ty) < a^d{Sx,Sy)+a2d{Sx,Tx)+a:id{Sy,Ty)+a^d{Sx,Ty)+a5d{Sy,Tx),*

where $\overset{5}{\underset{t=i}{\wedge}}$ a, < 1.

(di3) *d{Tx,Ty) < ^max{d{Sx,Sy),d{Sx,Tx),d{Sy,Ty),d{Sx,Ty),d{Sy,Tx)},*

where A; G (0,1).

PROOF. The proof of the Corollary 4.3.1 follows from Theorem 4.3.1 and Examples 4.2.1-4.2.10.

REMARK 4.3.2. Corollaries corresponding to (^4) to (dis) are new results as they are free from any condition on containment of range of the involved mappings. Also, no existing result proved for four mappings can deduce them as corollaries due to obvious reason. Also notice that some of our corollaries are seeming new to the literature (e.g. corollary corresponding to *{d5,ds,dg* and rfia).

As an application of Theorem 4.3.1, we prove a common fixed point theorem for two finite families of mappings which runs as follows:

THEOREM 4.3.2. Let *{Si,* 52,..., *Sp}* and (Ti, T2, *...,Tm}he* two finite families of self mappings of a metric space *{X,d)* with *S =^ S1S2* . • • *Sp* and *T = T1T2* • --*Tm* satisfying property *{E.A)* and condition (4.1). If *S{X)* is a complete subspace of *X*, then

(^14) *{S, T)* has a point of coincidence.

Moreover, if *TiT^* = *TjTu* *SkSi* = *SiSk* and *TiSk* = *SkTi* for all *i,j* *e* *h* = {1,2,..., *m}* and A; / e /2 = {1,2,... ,p}, then (for alH € A and fc G *h) Sk* and *Ti* have a common fixed point.

PROOF. The conclusion *{du)* is immediate as 5 and *T* satisfy all the conditions of Theorem 4.3.1. Now appeahng to componentwise commutativity of various pairs, one can immediately prove that *TS* = *ST* and hence, obviously the pair (5, T) is weakly compatible. Note that all the conditions of Theorem 4.3.1 (for mappings *S* and *T)* are satisfied ensuring the existence of unique common fixed point, say *t.* Now one needs to show that *t* remains the fixed point of all the component mappings. For this consider

T(Tit) = ((Ti,T2,... *,Tm)Ti)t* = *{T,T2..* .T^_i)((T^TOt) = (Ti... *Tm-i){TiTmt)*

... = ... = *...^TiTi{T2T,T,...Tmt)* = *TiTi{T2T3...Tmt)^Ti{Tt)* = *Tit.*

Similarly, one can show that,

T{Skt) = *Sk{Tt)* = *Skt,* *SiSkt)* = *SkiSt)* = *Skt*

and *S{Tit)* = *Ti{St)* = *Tit*

which show that (for all *i* and *k) Tit* and *Skt* are other fixed points of the pair (5, *T).* Now appealing to the uniqueness of common fixed points of the pair separately, we get

t = *lit* =^ *Okt*

which shows that t is a common fixed point of *Ti* and *Sk* for all *i* and *k.*

By setting Ti = Ta = ... = T^ = F and 5i = ^2 = ... = 5p = B in Theorem 4.3.2, we deduce the following:

COROLLARY 4.3.2. Let F and B be self mappings of a metric spax^e *{X, d)* satisfying property *(E.A)* along with the inequahty (4.1) for all distinct *x,y* *e* *X.* If *B'"{X)* is a complete subspace of X, then F and *B* have a unique common fixed point provided *FB* = *BF.*

By setting % = F (for all *i)* and *Sk - Ix* (identity lisiapiJing) in Theorem 4/^.2, one deduces the following fixed point theorem which can be viewed as a jjafiant of Bryant's theorem (cf. [16]).

COROLLARY 4.3.3. Let F be a self mapping of a metric space *{X, d)* such that there exists some m G iV satisfying

F{d{F^x, F^y), d(x, y d(x, *F^x), d{y,* F"»y), *d{x, F^y), d{y, F^'x))* < 0,

for all *x,y e X* and *F e* ^- If *F^{X)* is a complete subspace of *X,* then F has a unique fixed point.

4.4. ILLUSTRATIVE EXAMPLES

In this section, we present two examples illustrating the results proved in Section 4.3. The following examples illustrate the validity of the hypotheses of Theorem 4.3.1 besides establishing its utility over earlier results due to Jungck [70] and others proved for a pair of mappings.

EXAMPLE 4.4.1. Consider X = [-1,1] with the usual metric. Define self mappings S and T on A" as in Example 4.1.1.

Notice that all the mappings are discontinuous even at their unique common fixed point '0' which is their common coincidence point as well. Also the pair (5, *T)* is commutative at coincidence point '0'. Define a continuous function F : 3f?^ -> 3? such that *F{ti,t2,...,te)* = ti - *kmQx{t2,h,U,\{t^ + te,)},* where *k e* (0,1), then one can verify that F satisfies Fj, *F^* and F3. By a routine calculation one can also show that inequahty (4.1) is satisfied for fc = ^.

Example 4.4.1 exhibits that Theorem 4.1.1 due to Jungck [70] cannot be used in this context as all the involved mappings are discontinuous whereas Theorem 4.1.1 requires the continuity of at least one of the involved mappings besides *T{X)* C *S{X)* which is not met in the present example.

Finally, we show that requirement of completeness of *S{X)* is essential in Theorem 4.3.1 and cannot be relaxed even if the space X is complete.

EXAMPLE **4.4.2.** Let *X* = {O, l, 1/2,1/2^, 1/23,...} be a metric space with the usual metric d(x,y) = *\x — y* for all a;,y € *X*. Define mappings *T,S : X* -^ *X \>y* r(0) = l/22,T{l/2'*) = l/2"+2^5(0) = 1/2 and 5(1/2") = 1/2"+^ for n = 0,1,2,... respectively. Clearly, pair (5,T) enjoys property *{E.A)* (e.g. *Xn* — ^). Define a continuous function F : 3?^ -)• 3? as follows

F{hM, ...M) = *tl- atl* - ^ r ^ ^ ^ > (with a = ^ and 6 = ^^

then *F* satisfies Fi,F2 and F3 (see [122]).

By a routine calculation one can verify that all the conditions of Theorem 4.3.1 are satisfied except the completeness of the subspaces *S{X)* and *T{X).* Note that *S* and *T* have no point of coincidence. Here it is fascinating to note that in the set up of Theorem 4.3.1 even the completeness of the space cannot ensure the existence of coincidence point as the space *X* is complete in the present example. Notice that *S* and *T* are not continuous at 0.

4.5. IMPLICIT FUNCTION II

In this section, we define a new class of implicit functions and furnish a variety of examples which include most of the well known contractions of the existing literature besides admitting several new ones. Here it is fascinating to note that some of the presented examples are of nonexpansive type (e.g. Examples 4.5.16 and 4.5.19) and Lipschitzian type (e.g. Examples 4.5.12, 4.5.14 and 4.5.15). Here, it may be pointed out that most of the following examples do not meet the requirements of implicit function due to Popa [122]. In order to describe the present implicit function, let $ be the family of lower semi-continuous functions F : 3?^ ->• 3? satisfying the following conditions.

(Fi) : F(t,0,t,0,0,t) > 0, for all *t* > 0,

(F2) : *F{t,0,0,t,t,0)* > 0, for all *t* > 0,

(F3) : *F{t, t,* 0,0, *t, t)* > 0, for all *t* > 0.

EXAMPLE 4.5.1. *De^ne F{ti,t2,---,k)* •^*l-^^as*

F(ti,t2,--- ,h) = *ti-* A;max{«2,^3,^4,^5,^6}, where *k* £ [0,1).

(Fi): *F{t*, 0, *t*, 0,0, *t)* = *t{l* - fc) > 0, for alH > 0,

(Fa): *F{t*, 0,0, *t*, *t*, 0) = *t(l* - fc) > 0, for all *t* > 0,

(F3): *F{t,t,0,0,t,t)* = f(l - fc) > 0, for aU <> 0.

EXAMPLE 4.5.2. DefineF(«i,f2,--,<6) :3?^->3fJas

F(ti,<2,--- ,<6) = *i -A;max{f2,<3,M5,M6}, where *k* e [0,1)-

(Fi): F(t, 0,t,0,0,t) = t(l - fc) > 0, for all *t* > 0,

(F2): F(t,0,0,f,t,0) = 0 0 , for all *t* > 0,

(F3): F(t,t,0,0,t,t) = «(l - fc) > 0, for all *t* > 0.

EXAMPLE 4.5.3. Define F(ti,t2,-•• .te) :9fi+->-31? as

F(ii,f2,-" *yh) = ti* - *k{max{tl,t3U,t5te,t3t5,t4te}]'^*, where fc G [0,1).

EXAMPLE 4.5.4. Define F(ii,t2,--- *,h):Rl^3tas*

F(f 1, f2, • • • , te) = ti-a[/3 max{t2, tg, t4, *h*, tej-f (l-/3)(max{t|, *tzU,^^*, *tste*, t^g})^],

where a G [0,1) and *P>0.*

EXAMPLE 4.5.5. Define F(<i,f2,--- *,k):^l^Ras*

F(ti, t2, • • • , te) = ti - amax{t^, t^, t^} - /?max{t3t5, ^4^6} - 7*5*6,

where a, /3,7 > 0 and a + 7 < 1.

EXAMPLE 4.5.6. Define F(ti,t2,--- *,te)-. Ul-^ U as*

F{ti,t2, ••• , ^e) = (1 + a*2)*i - *a* max{t3<4, *tste}* - *0* max{f2, *3, *U*, *h*, *te},*

where a > 0 and *(3* G [0,1).

EXAMPLE 4.5.7. Define F(ii,i2,--- *,te):Rl^^as*

F{t\M,--- M) = *h* -ai2-/?max{f3,<4} - 7max{t3 + ^4,^5 + ^e},

where a, y9,7 > 0 and a + /9 + 27 < 1.

EXAMPLE 4.5.8. *De6neF(tut2,-'-,te):^+-^^as*

$$F\{ti,t2,\text{-- • .te}) = ti - (\hat{}(\max\{t2,t3,t4,t5,t6\}),$$

where (^ : J?"' —>^ 3? is an upper semi-continuous function such that (^(0) = 0 and ***4){t) < t* for all *t* > 0.**

EXAMPLE 4.5.9. ***De&neF{ti,t2,---,te):^+-^^as***

$$F\{ti, \hat{}2, \bullet\bullet\bullet . \hat{}e) = *i — <\hat{}(*2, \hat{}37\ U, h, h),$$

where 0 : 3?^ —>• 3? is an upper semi-continuous and nondecreasing function in each coordinate variable such that *^{t, t,at, bt, ct) <t* for each t > 0 and a, 5, c > 0 with ***a + b + c<3.***

EXAMPLE 4.5.10. DefineF(ti,f2,---.^e)--SR^->3ft as

$$F\{ti,t2, \bullet" \hat{}to) = ty — (f>(t2, \hat{}4)\ Me, \hat{}3\hat{}65\ \hat{}4\hat{}5),$$

where 0 : 9f)^ —> 3? is an upper semi-continuous and nondecreasing function in each coordinate variable such that *4>it, t, at, bt, ct) < t* for each *t > 0* and a, 6, c > 0 with ***a + b + c<S.***

EXAMPLE 4.5.11. Define F(ii,f2,-•• ,^e) : 3f?^ ->• S? as

$$Fitut2,-,t,) = h\hat{}\text{-}\hat{}'\hat{}\text{-}\hat{}'\hat{}r'\hat{}''\text{-}'''\hat{}\hat{} \quad \text{if} \quad \hat{}3+\hat{}4\hat{}0$$
$$[\ ti, \qquad \text{if} \quad t3+t4 = 0,$$

where a, /5,7 > 0 and /3 + 7 < 1.

EXAMPLE 4.5.12. Define F(fi,f2,-•• *,te):^l^Uas*

$$F\{t,,t2,\text{---} \quad ,te)=< \begin{matrix} tP.ktP\text{-}\hat{}A\pm\hat{}, & \text{if} \quad \hat{}3 + \hat{}47\hat{}0 \\ C3+H & \\ \hat{}\ ti, & \text{if} \quad tz + ti = 0, \end{matrix}$$

where p > 1 and *k>0.*

EXAMPLE 4.5.13. Define F(ti,fa,-•• ,<6) :3?+^3fJas

$$i'[ti,t2,\text{-} \bullet\bullet ,te) — \backslash \quad \frac{4 + ti}{ts + te}$$
$$ti, \qquad \text{if} \quad ts + te\hat{}O,$$

where Q, /?, 7 > 0 and /? + 7 < 1.

EXAMPLE 4.5.14. Define F(ti,f2,-" .te) : 31^-^3? as

[h, *a h + te = o,*

where fc > 0.

EXAMPLE 4.5.15. Define F(ti,t2,'•• .te) ••3i^+->3f? as

[ti, if *h+t4 = 0oTh + t6^0,*

where A; > 0.

EXAMPLE 4.5.16. Define F(ti,t2,-•• .te) :3fit ^ 3ft as

EXAMPLE 4.5.17. Define F(ti,t2,-•• .^e) ••9^^-^ 3ft as

i + 15^6

EXAMPLE 4.5.18. Define F(ti,t2,--- .te)-.3?^-^ 3ft as

F{tut2. ••• ,te) = tl- atl - ^^._^^2^^2.

where a, /? > 0 and *a + P < 1.*

EXAMPLE 4.5.19. Define F(ii,f2,-•• .ie) :3ft^ ^3ftas

f2f2 , f2f2

EXAMPLE 4.5.20. Define F(ii,f2,-" ,ie) : 3ft^-> 3ft as

Fih, t2,--- , te) = tl- atlti - PUUU - itlu - vtstl

where a, /5,7,7; > 0 and a + 7 + r/ < 1.

Since verification of requirements (Fi,F2 and F3) for Examples 4.5.3-4.5.20 are easy, hence details are not included.

4.6. RESULTS **VIA COMMON PROPERTY (E.A)**

We begin with the following observation.

LEMMA 4.6.1. Let *A, B, S* and *T* be self mappings of a metric space *{X, d)* such that

(dis) the pair *{A,S)* (or *{B,T))* satisfies the property *(E.A),*

(die) *A{X)* C *T{X)* (or *B{X)* C *S{X)),*

(dn) for all *x,y e X* and F G $

F{d{Ax,By),d{Sx,Ty),d{Ax,Sx),d{By,Ty),d{Sx,By),d{Ty,Ax)) < 0. (4.2)

Then the pairs *{A,S)* and *{B,T)* satisfy the common property (^..4).

PROOF, if the pair *{A,S)* enjoys property *(E.A),* then there exists a sequence *{xn}* in X such that

hm *Axn* = hm *Sxn* = *t,* for some *t E X.*
n—>oo Ti->cx)

Since *A{X)* C *T{X),* hence for each {x„} there exists {y„} in *X* such that Ax,, = *Tyn-* Therefore, lim Ty„ = Um *Axn* = <• Thus, in all we have *Axn* -^ *t,Sxn* -^ *t*
n->oo n—>oo
and *Tyn* -^ *t.* Now, we assert that 5y„ -^ f. If not, then using (4.2), we have

*F{d{Axn, Byn), d{Sxn, Ty^l*d(Aa:„, 5x„), d(5y„, Ty^), d(5x„, %„), d(Tyn, ^a;„)) < 0

which on making n —> co, reduces to

F{d{t, Byn), 0,0, d(By„, *t), d{t,* J3y„), 0) < 0

a contradiction to *{F2).* Hence *Byn* —> *t* which shows that the pairs *{A,S)* and *{B,T)* satisfy the common property *(E.A).*

REMARK 4.6.1. The converse of Lemma 4.6.1 is not true in general. For a counter example, one can see Example 4.7.1.

Now, we state and prove our main result for two pairs of weakly compatible mappings satisfying an implicit function.

THEOREM 4.6.1. Let *A, B, S* and *T* be self mappings of a metric space (X, d) which satisfy inequaUty (4.2). Suppose that (dis) the pairs *{A, S)* and *{B,T)* enjoy the common property *[E.A),*

{dig) *S{X)* and *T{X)* are closed subsets of *X.*

Then the pair *{A,S)* as well as *{B,T)* have a coincidence point. Moreover, *A, B, S* and *T* have a unique common fixed point provided both the pairs *{A, S)* and *{B, T)* are weakly compatible.

PROOF, since the pairs *{A,S)* and (5,T) enjoy common property *{E.A),* then there exist two sequences {x„} and {y„} in *X* such that

$$\lim_{n\rightarrow oo} Axn = \lim_{n\rightarrow oo} Sxn = \lim_{n\rightarrow oo} \text{By„} = \lim_{n\rightarrow cx)} Tyn = t, \qquad \text{for some t € X.}$$

If *S{X)* is a closed subset *oi X,* then $\lim_{n\rightarrow oo}$ 52;„ = < 6 *S{X).* Therefore, there exists a point *u e X* such that *Su — t.* Now we assert that *Au* = *Su.* If not, then using (4.2), we have

$$F\{d\{Au, Byn), d\{Su, Ty\hat{}), d\{Au, Su), d\{Byn, \text{Ty„}), d\{Su, By„), d\{Tyn, Au))<0$$

which on making n —> oo, reduces to

$$F\{d\{Au, t), d\{Su, t), d\{Au, Su), d\{t, t),d\{Su, t), d\{t, Au)) < 0$$

$$\text{or} \qquad F\{d\{Au, Su), 0, d\{Au, Su), 0,0, d\{Su, Au)) < 0$$

a contradiction to (Fi). Hence *Au* = *Su.* Therefore, u is a coincidence point of the pair *{A,S).*

If *T{X)* is a closed subset of *X,* then $\lim_{n\rightarrow oo}$ *Tyn* = *t e T{X).* Therefore, there exists a point *w E X* such that *Tw* = *t.* Now we assert that Bio = *Tw.* If not, then again using (4.2), we have

$$F\{d\{Axn, Bw), d\{Sxn, Tw), d\{Axn, Sxn), d\{Bw, Tw),d\{Sxn, Bw), d\{Tw, Axn)) < 0$$

which on making *n -¥ oo,* reduces to

$$F\{d\{t, Bw), d\{t,Tw), d\{t, t), d\{Bw, Tw), d\{t, Bw),d\{Tw, t)) < 0$$

$$\text{or} \qquad F\{d\{Tw, Bw), 0,0, d\{Bw, Tw), d\{Tw, Bw), 0) < 0$$

a contradiction to (F2). Hence $Bw = Tw$, which shows that lo is a coincidence point of the pair (B,T).

Since the pair $\{A, S)$ is weakly compatible and $Au = Su$, hence $At = ASu = SAu = St$. Now we assert that t is a conunon fixed point of the pair (A, $S)$. Suppose that $At \wedge t$, then using (4.2), we have

$$F\{d\{At, Bw), d\{St,Tw), d\{At, St), d\{Bw,Tw), d\{St, Bw), d\{Tw, \mathrm{At})) < 0$$

$$\text{or} \qquad F\{d\{At, t), d\{At, t), 0,0, d\{At, t), d\{t. At)) < 0$$

a contradiction to $[Fz)$.

Also the pair $(5, T)$ is weakly compatible and $Bw — Tw$, then $Bt = BTw = TBw = Tt$. Suppose that $Bt \wedge t$, then using (4.2), we get

$$F\{d\{Au,Bt),d\{Su,Tt),d\{Au,Su),d\{m,Tt),d\{Su,Bt),d\{Tt, \quad \mathrm{Au})) \quad < \quad 0$$

$$\text{or} \qquad F\{d\{Bt, t), d\{Bt, t), 0,0, d\{Bt, t), d\{t, Bt))<0$$

a contradiction to (F3). Therefore, $Bt = t$ which shows that t is a common fixed point of the pair $\{B,T)$. Hence t is a common fixed point of both the pairs (A,5") and $\{B,T)$. Uniqueness of common fixed point is an easy consequence of inequality (4.2) (in view of condition (F3)). This completes the proof.

THEOREM 4.6.2. The conclusions of Theorem 4.6.1 remain true if the condition *(dig)* of Theorem 4.6.1 is replaced by the following.

(rf2o) $A\{X)$ C $T\{X)$ and $B\{X)$ C $S\{X)$.

As a corollary of Theorem 4.6.2, we can have the following result which is also a variant of Theorem 4.6.1.

COROLLARY 4.6.1. The conclusions of Theorems 4.6.1 and 4.6.2 remain true if the conditions *(dig)* and (^20) are replaced by the following.

(rf2i) $A(X)$ and $B\{X)$ are closed subsets of X provided $A(X)$ C $T\{X)$ and $B\{X)$ C $S\{X)$.

THEOREM 4.6.3. Let A,5,5 and T be self mappings of a metric space $\{X, d)$ satisfying inequality (4.2). Suppose that (d22) the pair *{A,S)* (or (B,T)) has property *(E.A),*

(^23) *A{X)* c *T{X)* (or B(X) c *S{X)),*

(d24) S(X) (or *T{X))* is a closed subset of *X.*

Then the pair *{A, S)* as well as (B, T) have a coincidence point each. If the pairs (A, *S)* and (B, T) are weakly compatible, then *A, B, S* and *T* have a unique common fixed point.

PROOF, in view of Lemma 4.6.1, the pairs *(A^S)* and *(B,T)* satisfy the common property (£^.^4), i.e. there exist two sequences {x„} and *{yn}* in *X* such that

$$\lim_{n \to oo} Axn = \text{hm}_{n \to oo} 5x_„ = \lim_{n \text{—}foo} By_„ = \lim_{n \to oo} Ty_„ = t\ E\ X.$$

If *S{X)* is a closed subset of *X,* then on the lines of Theorem 4.6.1, the pair *{A, S)* has coincidence point, say u, i.e. *Au* = *Su.* Since -(4(X) C *T{X)* and >IM e A(X), there exists *w* £ *X* such that *Au* = Tiu. Now we assert that *Bw* = *Tw.* If not, then using (4.2), we have

$$F\{d\{Axn, Bw), d\{Sxn, Tw), d\{Axn, Sxn), d\{Bw, Tw), d(Sxn, Bw), d(Tw, Axn)) < 0$$

which on making n -> oo, reduces to

$$F\{d\{t, Bw),d\{t, Tw), d\{t, t), d\{Bw, Tw), d\{t, Bw), d\{Tw, t)) < 0$$

$$\text{or} \qquad F(d\{Tw, Bw), 0,0, d\{Bw,Tw), d\{Tw, Bw), 0) < 0$$

a contradiction to (F2). Hence *Bw* = *Tw,* which shows that w is a coincidence point of the pair *{B,T).* Rest of the proof can be completed on the lines of the proof of Theorem 4.6.1.

By choosing *A, B, S* and *T* suitably, one can deduce coroUaxies for a pair as well as for a triod of mappings. The detail of two possible corollaries for a triod of mappings are not included. As a sample, we outline the following natural result for a pair of self mappings.

COROLLARY 4.6.2. Let A and 5 be self mappings of a metric space *(X,d).* Suppose that

(^25) the pair (.4,5) has property *(E.A),*

(^26) for all *x,yeX* and F G $

F(diAx, Ay), diSx, Sy), d{Ax, Sx), d{Ay, 5y), *d{Sx, Ay),* d(5y, Ax)) < 0 (4.3)

(d27) *S{X)* is a closed subset of *X.*

Then *A* and *S* have a coincidence point. Moreover, if the pair (^4,5) is weakly compatible, then *A* and *S* have a unique common fixed point.

COROLLARY 4.6.3. The conclusions of Theorem 4.6.1 remain true if inequality (4.2) is replaced by one of the following contraction conditions. For *eHx,y e X,*

(dss) *d{Ax, By)* < *k*max{d(5x,*Ty),d{Ax,Sx),d{By,Ty),d{Sx, By),d{Ty, Ax)},*

where fc e [0,1).

(^29) *d{Ax, By)* < *kmax{d{Sx, Ty), d{Ax, Sx),d(Ax, Sx)d{Sx, By), d{By,Ty)*

d{Ty, Ax)}, where *k e* [0,1).

(dso) *d{Ax, By)* < *k[max{d'^{Sx, Ty), d{Ax, Sx)d{By, Ty), d{Sx, By)d{Ty, Ax),*

diAx, Sx)d{Sx, By), d{By, Ty)d{Ty, Ax)} 5,

where *k* G [0,1).

(rfai) *d{Ax,By)* < *a[Pmax{d{Sx,Ty),d{Ax,Sx),d{By,Ty),d{Sx,By),d{Ty,Ax)}*

+(1 - *p){m&x{d\Sx, Ty),d{Ax, Sx)d{By, Ty), d{Sx, By)d{Ty, Ax),*

diAx, Sx)d{Ty, Ax), d{By, Ty)d{Sx, By)})^,

where *a* G [0,1) and /? > 0.

(^32) *d^{Ax,By)* < *Qmax{d'^{Sx,Ty),d'^{Ax,Sx),d'^{By,Ty)}* + *Pmax{d{Ax,Sx)*

d{Sx, By),d{By, Ty)d{Ty, Ax)} + *^d{Sx, By)d{Ty, Ax),*

where a, ^, 7 > 0 and a + 7 < 1.

(43) *{l+ad{Sx,Ty))d{Ax,By)* < *amax{d{Ax,Sx)d{By,Ty),d{Sx,By)d{Ty,Ax)}*

+P max{d{Sx, Ty), d{Ax, Sx), d{By, Ty),d{Sx, By), d{Ty, Ax)},

where a > 0 and/9 G [0,1).

(^34) d{Ax, By) < ad{Sx, Ty) + /3max{d(ylx, Sx), d{By,Ty)} + 7 max{d(Ax, Sx)

+d{By, Ty), d{Sx, By) + d{Ty, Ax)},

where a, ^, 7 > 0 and a + /3 + 27 < 1.

(das) d{Ax, By) < <f>{max{d{Sx,Ty), d{Ax, Sx),d{By,Ty), d{Sx, By),d{Ty, Ax)}),

where (^ : S?*' ->• 3? is an upper semi-continuous function such that (f)(0) — 0 and <j)it) < t for all t > 0.

(dae) d{Ax,By) < (}>{d{Sx,Ty),d{Ax,Sx),d{By,Ty),d{Sx,By),d{Ty,Ax)),

where 0 : 9ft^ —>• 3? is an upper semi-continuous and nondecreasing function in each coordinate variable such that (f)(t, t, at, bt, ct) < t for each t > 0 and o, 6, c > 0 with a + b + c<3.

(da?) d\Ax, By) < (p{d^{Sx, Ty), d{Ax, Sx)d{By, Ty),d{Sx, By)d{Ty, Ax),

d{Ax, Sx)d{Ty, Ax),d{By, Ty)d{Sx, By)),

where (j!>: 3?^ -> 3!? is an upper semi-continuous and nondecreasing function in each coordinate variable such that (p{t,t,at,bt,ct) < t for each t > 0 and a,b,c> 0 with a + b + c<Z.

In following contraction conditions, we denote D = d{Ax, Sx) + d{By, Ty) and Di=d{Sx,By) + d{Ty,Ax).

.(f{Ax,Sx) + d'^{By,Ty)

'''^^^''^y^^^-d{Ax,Sx) + diBy,Ty)

{d3s)d{Ax,By)< < +j{d{Sx,By) + d{Ty,Ax)), if D ^ 0

^0, if D = 0,

where a, /?, 7 > 0 and /3 -f 7 < 1.

' kdP(Sx,Ty)+

djAx,Sx)dP{By,Ty) + d{Sx,By)dP{Ty,Ax) .. ^ , ^

(dag) dF{Ax, By)<l d{Ax, Sx) + d{By, Ty) n u^u

^0, if £> = 0,

wherep > l and fc > 0.

(rf4o) *d{Ax, By)* < *i* *adlSx Tv)* + *a^iSx,By)* + *d^{Ty,Ax)* / *ad{Sx,Ty)* + *P* *^^Sx,By)* + *d{Ty,Ax)* +'*ridiAx,Sx)* + *diBy,Ty)),* if *Di^O*

l0. if Di = 0,

where a, /3,7 > 0 and /? + 7 < 1.

(d4i) *d{Ax, By)* < *i* ' *kd{Sx,Ty)* *d{Ax, Sx)d{By, Ty)* + *d{Sx, By)d{Ty, Ax)* / *d{Sx,By)* + *d{Ty,Ax)* if Di 7^ 0

0, if Di = 0,

where A; > 0.

(^42) *d{Ax, By)* < *i* *kd{Sx, Ty)* + *d{Ax, Sx)d{By,Ty)* + *d(Sx, By)d{Ty, Ax)* / *d{Ax, Sx)* + *d(By, Ty)*

, *diAx, Sx)diSx, By)* + *d{By,Ty)d{Ty,Ax)"* / *d{Sx,By)* + *d{Ty,Ax)* jf ^ _^ Q ^^ ^ Q

[0. if D = 0 or Di = 0,

where A; > 0.

(rl \^*(A.* D.X ^ *d{Ax,Sx)d{By,Ty)+d{Sx,By)d{Ty,Ax)* / 1 + *d{Sx, Ty)*

where a,/? G [0,1).

(d45) *d\Ax,By)* < *Qd\Sx,Ty)+P* rf(gx, *By)d{Ty, Ax)* / *l* + *d^{Ax,Sx)* + *d^{By,Tyy*

where *a,l3 >Q* and a + y5 < 1.

(d46) rf^C^lx, *By)* < *<f{Ax, Sx)d?{By, Ty)* + *d\Sx, By)d\Ty, Ax)* / \ + *d{Sx,Ty)*

(^47) *d^{Ax, By)* < *ad'^{Ax, By)d{Sx, Ty)+pd{Ax, By)d{Ax, Sx)d{By, Ty)*

+ *jd^Sx, By)d{Ty, Ax)* + 7?rf(5x, *By)d\Ty, Ax),*

where a, 7,77, y8 > 0 and a + 7 + 77 < 1.

PROOF. Proof follows from Theorem 4.6.1 and Examples 4.5.1-4.5.20.

REMARK 4.6.2. Corollaries corresponding to contraction conditions (^28) to (^47) are new results as these never require aaiy conditions on containment of ranges. Some contraction conditions (e.g. ^28, c^si, ^33—^42) ui above corollary are well known and generalized relevant results from [2,21,23-25,34,36,46,50,53,57-59,61,69,72,83,89, 94,99,105,138,145] while some others are new ones (e.g. ^29,^30,^32,^43 - ^47)-

As an application of Theorem 4.6.1, we have the following result for four finite families of self mappings.

THEOREM 4.6.4. Let {Ai,A2,...,>!„.}, {Bi,B2,...,Bp}, {5i,52,...,5n}and *{Ti,T2,... ,Tq}he* four finite famiUes of self mappings of a metric space *{X, d)* with *A* = Ai A2 *...Am, B = B1B2 ...Bp, S = S1S2,...* 5„ and T = T1T2... T, satisfying condition (4.2),

(^43) the pairs (^4,5") and *(B, T)* share common property *{E.A),*

(^49) *S{X)* and *T{X)* are closed subsets of *X.*

Then the pair *{A,S)* as well as *{B,T)* has a coincidence point.

Moreover, iiAiAj = AjAi, BkBi = BiBk, SrSs = SsSr, TtTu = TuTu ABk = BkAi and *SrTt = TtSr* for all *i,j* € /j = {1,2,..., m}, *k,leh* = {1,2,... ,p}, r, s 6 /s = (1,2,...,n} and i,u e A = {1,2,...,q}, then (for all *lE Ii,k e h^r e h* and *t* e *I^) Ai,Bk,Sr* and *Tt* have a common fixed point.

PROOF. Proof follows on the fines of a result due to Imdad et al. [63, Theorem 2.2].

By setting *Ai = A2 = ... = Am = G, Bi = B2 = ...-- Bp = H, Si* = 52 = ... = Sn = / and Ti = T2 = ... = Tq = J in Theorem 4.6.4, we deduce the following.

COROLLARY 4.6.4. Let *G, H, I* and *J* be self mappings of a metric space *{X, d)* such that the pairs (G"*, /") and *{H^, J^)* have common property (£^.^4) and also satisfy the condition

FidiG'^x, H^y), dirx, J^y), d(G""x, *Px), d{HPy, J^y),*

$$d(\Gamma x, JP^q y), d(J^q y, (\Gamma x)) < 0,$$

for all $x, y \in X$ and $F \in G$ $ where m, n, p and q are fixed positive integers. If $P\{X)$ and J^(X) are closed subsets of X, then G, H, I and J have a unique common fixed point provided $GI = IG$ and HJ = Jif.

REMARK 4.6.3. By restricting four families as $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{S_1\}$ and $\{T_1\}$ in Theorem 4.6.4, we deduce a substantial but partial generalization of the main results of Imdad and Khan [57,58] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and weakening completeness requirement of the space to the closedness of subspaces.

REMARK 4.6.4. Corollary 4.6.4 is a slight but partial generalization of Theorem 4.6.1 as the conuuutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are stronger as compared to weak compatibility in Theorem 4.6.1.

REMARK 4.6.5. Results similar to Corollary 4.6.3 can be derived from Theorems 4.6.2-4.6.3 and Corollaries 4.6.2 and 4.6.4. For the sake of brevity, we have not included the details.

4.7. ILLUSTRATIVE EXAMPLES

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results presented in Section 4.6 over the majority of previously known results proved till date with some possible exceptions.

EXAMPLE 4.7.1. Consider $X = [-1,1]$ equipped with the usual metric. Define self mappings A, B, S and T on X as

$$A(-1) = A1 = 3/5, \quad Ax = x/4, \quad -1 < x < 1,$$
$$B(-1) = B1 = 3/5, \quad Bx = -x/4, \quad -1 < x < 1,$$
$$S(-1) = 1/2, \quad Sx = x/2, \quad -1 < x < 1, \text{ and } S1 = -1/2, \text{ and}$$
$$T(-1) = -1/2, \quad Tx = -x/2, \quad -1 < x < 1, \text{ and } T1 = 1/2.$$

Consider sequences $\{x_n = \frac{1}{n}\}$ and $\{y_n = \frac{1}{n}\}$ in X. Clearly,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0$$

which shows that pairs *{A,S)* and *{B,T)* satisfy the common property *(E.A).* Define a continuous implicit function F : 3ft^ -^ ^ such that *F{t-[,t2,...,h)* — *ti-k* max{t2, *h, U, h, tf}* where *k* € [0,1) and F 6 $. By a routine calculation, one can verify the inequaUty (4.2) with ib = i. Also, *A{X)* = *B(X)* = {§} U *{=f, \)* ^ *S{X)* = *T{X)* = *[=^,* |] . Therefore, all the conditions of Theorem 4.6.1 are satisfied and 0 is a unique common fixed point of the pairs (-4,5) and *{B,T)* which is their coincidence point as well.

Here it is worth noting that none of the theorems (with some possible exceptions) can be used in the context of this example as Theorem 4.6.1 never requires any condition on the containment of ranges of the involved mappings while the completeness condition is replaced by the closedness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed whereas all earlier theorems (prior to 1997) require the continuity of at least one involved mapping.

Now, we furnish an example which presents a situation applicable to Theorems 4.6.1, 4.6.2 and 4.6.3.

EXAMPLE 4,7.2. Consider *X* = [2,20] equipped with the usual metric. Define self mappings *A,B,S* and T on X as

Ax = *2,* *xe* {2} U (5,20], Ax - 4, 2 < x < 5,

Bx = *2,* *xe* {2} U (5,20], Sx = 3, 2 < x < 5,

S2 = 2, 5x = 8, 2 < X < 5, *Sx* = *{x* + l)/3 , x > 5 and

r2 = 2, Tx = 12 -I- X, 2 < X < 5, Tx = x - 3, x > 5.

Clearly, both the pairs *iA,S)* and (5,r) satisfy the common property (£^..4) as there exist two sequences {x„ — *5* + *^}*, {y„ = *5* + *^} C X* such that

$$\lim_{n—¥oo} Axji — \lim_{n—¥oo} 5x„ = \lim_{n—>oo} Byn = \lim_{n-»oo} Tyn = 2.$$

Also *A{X)* = {2,4} C [2,17] = *T{X)* and *B{X)* = {2,3} C [2,7] U {8} = *S{X).* Define *F{ti,t2, ...,te)* *:Ul* ^^ with ^3 + ^4 7^ 0 as

$$F{tut2,...,te) = ti-at2~P \frac{tl + tl}{h + U} - ih + te]$$

where a, /3,7 > 0 with at least one is nonzero and /3 -I- 7 < 1

By a routine calculation one can verify that contraction condition (4.2) is satisfied for a = 7 = i and *P* = *i*- If x, y 6 {2} U (5,20], then *d{Ax, By)* = 0 and verification is trivial. If x e (2,5] and *y* > *5,* then

$$ad\{Sx,Ty)+P \quad \frac{<f\{Sx,Ax) \; + \; <P\{Ty,By)}{\_ \;\; d\{Sx,Ax) \; + \; d\{Ty,By)} \; + \; j[d\{Sx,By) \; + \; d\{Ty,Ax)]$$

$$= \qquad \backslash\backslash y\text{-}n\backslash\wedge\backslash \quad \frac{4\wedge \; + \; \backslash y\text{-} \; 5p}{,\; 4 + |y - 5| \;\; J} \; + \; \wedge[6 \; + \; |y-71]$$

r 1 + J(24 - 2y) > 2 = *d{Ax, By),* if y e (5,7]

^{ *l+'i* = *%>2* = *d{Ax,By),* ifye(7,11]

^ I + i(2y - 12) > 2 = *d{Ax, By),* if y > U.

Similarly, one can verify the other cases. One may note that the pairs *{A, S)* and *{B,T)* commute at 2 which is their common coincidence point. All the needed pair-wise commutativity at coincidence point 2 are immediate. Thus all the conditions of Theorems 4.6.1, 4.6.2 and 4.6.3 are satisfied and 2 is the unique common fixed point *of A,B,S* and *T.* Here one may notice that all the mappings in this example are even discontinuous at their unique common fixed point 2.

Example 4.7.2 may lead an impression that Theorems 4.6.1, 4.6.2 and 4.6.3 are not different results. In what follows, we show that these results can be situationally useful, i.e. there do exist situations when one theorem is appUcable whereas others are not. In order to substantiate this view point, we furnish the following examples.

EXAMPLE 4.7.3. in the setting of Example 4.7.2 retain the same *A,B,T* and implicit function *F* and modify *S* as follows.

S2 = 2, S20 = 2, *Sx* = *S,* *2<x<* 5, *Sx* = *{x* + l)/3, 5 < x < 20.

Clearly, *S{X)* = [2,7) U {8} which is not a closed subset of *X.* Here, Theorems 4.6.2 and 4.6.3 are applicable but not Theorem 4.6.1.

EXAMPLE *4.7A.* in the setting of Example 4.7.2 retain the same *A,B* and implicit function *F* and modify *S* and *T* as follows.

$$52 = 2, \quad 520 = 2, \quad Sx = 8, \quad 2 < x < 5, \quad Sx = \{x + l)/3, \quad 5 < x < 20,$$
$$r2 = 2, \quad T20 = 2, \quad Tx = 12 + X, \quad 2 < x < 5, \quad Tx = x - 3, \quad 5 < x < 20.$$

Clearly, *S{X)* = [2,7) U {8} and *T{X)* = [2,17) which are not closed subsets of *X*. Here Theorem 4.6.2 is applicable but not Theorems 4.6.1 and 4.6.3.

Chapter 5

FIXED POINTS OF SINGLE-VALUED AND HYBRID MAPPINGS VIA IMPLICIT FUNCTIONS

5.1. INTRODUCTION

Banach contraction principle has been extended and generalized in several ways which include a noted and significant generalization due to Edelstein [28]. The paper due to Edelstein [28] has inspired vigorous research activities since its appearance.

As pointed out in Chapters 2 and 3, contractive conditions do not ensure the existence of fixed points unless the space is assumed compact (cf. [28]) or the contractive condition is replaced by a relatively stronger condition. In recent years, the notion of noncompatible mappings have made it possible to prove results on strict contraction beyond compact metric spaces. The study of common fixed points of noncompatible mappings is a subject of investigation in the recent past which continues to be an interesting aspect for further study. Among all common fixed point theorems from existing literature on noncompatible mappings, the results contained in Pant [102,107] deserve special attention wherein author has shown the existence of common fixed points of strict contractions whenever the underlying space is not essentially compact. Most recently, Aamri and Moutawakil [1] and Liu et al. [94] introduced the notions of property *(E.A)* and common property *{E.A)* which are generalizations of compatible as well as noncompatible mappings and proved some common fixed point theorems for strict contraction in metric spaces. In previous chapters, we also studied certain aspects of property *(E.A)* (also common property *(E.A))* and proved some results for pairs of single-valued and hybrid mappings in semi-metric spaces as well as in metric spaces.

On the other hand, Nadler [101] extended Banach contraction principle to multi-valued mappings which is generally referred as Nadler contraction principle. At the same time, Markin [95] also extended Banach contraction principle to multi-valued

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mappings. Nadler fixed point theorem has also been extended by various authors to more general contraction conditions. To mention a few, we can cite [85,95,101]. On the other hand, Kaneko and Sessa [85] proved a fixed point theorem for generaUzed contraction involving a hybrid pair of compatible mappings [see 117,137,139]. This work due to Kaneko and Sessa [85] has inspired many authors to prove extensions and generalizations of their results. For the work of this kind, one can be referred to Pathak [117] and several others.

As pointed earher, Rhoades [126,128] carried out an exhaustive comparative study of contraction conditions wherein he introduced some contraction conditions and also established the equivalence of several contraction conditions. In recent years, some authors (e.g. [63,123,124]) utilized implicit functions instead of contraction conditions to prove common fixed point theorems. Implicit functions are proving fruitful due to their unifying power besides admitting new contraction conditions. In Chapter 4, we have also proved some results on common fixed points of self mappings using imphcit functions. In Section 5.2, we introduce a new class of implicit functions and utilize the same to prove our results because of their versatility of deducing several known and unknown contraction conditions in one go.

The aim of this chapter is to define a new family of implicit functions. In Sections 5.3 and 5.6, we also prove some general common fixed point theorems for single-valued and a sequence of hybrid mappings using implicit functions (defined in Sections 5.2 and 5.5). One may observe that the common property *(E.A)* relaxes the requirement of compactness of the underlying space and buys the required containment of ranges of involved mappings up to two pairs. In process, several known and unknown theorems improved and generaUzed. Some related results are also derived besides furnishing some illustrative examples.

5.2. IMPLICIT FUNCTION I

We now introduce a new class of implicit functions which is different from the one considered in Popa [122,124] and also the one introduced in Section 4.5. To describe it, let Q be the family of lower semi-continuous functions F : 3fJ^ -> 3? satisfying the following conditions.

(Fi) : *F{t,* 0,0, *t, t,* 0) > 0, **for** alH > 0,

(F2): *Fit,* 0, *t,* 0,0, *t)* > 0, **for** all *t* > 0,

(F3): F(i, *t,* 0,0, t,«) > 0, **for** aU <> 0.

EXAMPLE 5.2.1. *De6neF{ti,t2,---,t6)--^l^^as*

FihM,--- .tej^ti-maxUa, 2 ' 2 /'

(Fi): F(t, 0,0, t, t, 0) = ^ > 0, for alH > 0,

(F2): F(t,0,t,0,0,t) = I > 0, for allt > 0,

(F3): *F{t,t,0,0,t,t)* = 0, for all *t* > 0.

EXAMPLE 5.2.2. DefineF(ti,t2,---,*6) :3*+-^3ftas

F(ti, ^2, • • • 1 ^e) = ii - max (ta, ^3^5, *Uh* } •

(Fi): *F{t,* 0,0, i, i,0) = f > 0, for all *t* > 0,

(F2): *F{t,0,t,0,(i,t)=t>* 0, for all *t* > 0,

(F3): F(i, *t,* 0,0, ^ «) = 0, for all *t* > 0.

EXAMPLE 5.2.3. Define F(fi,t2,-••-^e)--SR^ ^3fi as

F(ii, ^2, • • • , ie) = ^1 - «max{i2, ^3, ^4} *-Praaxitsts, UU}* - ^*hU,*

where a,/?,7 > 0, a < 1 and a + 7 < 1.

EXAMPLE 5.2.4. DefineF(ti,f2,---,*6) ••5R+-^S^as

f ^3 + ^4 ^S + ^el

F(<i,<2, • • • ,^6) = (1 + "^2)^1 - *amsx{hUMh}* - max *i* ^2, — 2 ~ ' 2 J '

where a > 0.

EXAMPLE 5.2.5. Define F(ti,t2,-•• ,*6) :3^t ^3f? as

, , f, ^3 + ^4 *h* + *U* *h* + *h* *U±h*

F{tu «2, • • • , *k)* = «i - max <^ «2, —2—' —2—' —2—' —*T~* J '

EXAMPLE 5.2.6. Define F(ti,t2,-•• .^e) :3R^-^3fi as

F{iu^2, • • • , *te)* = *ti-a[pmax{t2,h, U, -(t5+t6)}+{l-P)[max{tl, hU, t^te, hh, Uh]\^,* where a G (0,1) and 0 < jS < 1.

EXAMPLE 5.2.7. *DehneF{ti,t2,---,h)-^+-^^Sis*

$$F\{tut2, \bullet\bullet\bullet\ ,tQ) = tl- atlh - PhhU - liU - Vhtl$$

where *a,0,'y,T]>O* and *a + y + T]* < 1.

EXAMPLE 5.2.8. *De&neF{tut2,--- ,k):^+-^Ras*

EXAMPLE 5.2.9. Define F(fi,<2,--- *,k)* :3?^-^Kas

where Q, /? > 0 and a + /? < L

EXAMPLE 5.2.10. DefineF(ti,t2,---.te) • 9f?+->Kas

$$-\hat{}(\hat{}1, \hat{}2, \bullet\bullet\bullet , \hat{}e) = \hat{}1 - <\hat{} (\max \{\hat{}2, <3, <4, <5, \hat{}e\}),$$

where 0 : 5?"*' —^ 3? is an upper semi-continuous function such that 0(0) = 0 and 0(f) < *t* for all *t* > 0.

EXAMPLE 5.2.11. DefineF(<i,t2,---.te) :9?+-^3?as

$$F(«i,f2,--- .\hat{}e) = \hat{}1 - 0(\hat{}2,\hat{}3,\hat{}4,\hat{}5,4),$$

where 0 : 3f?^ -)• Sf? is an upper semi-continuous and nondecreasing function in each coordinate variable such that *(f){t,tyat,bt,ct)* < *t* for each ^ > 0 and a,6,c > 0 with *a + h + c<?,*.

EXAMPLE 5.2.12. DefineF(ii,f2,---,<6) :3^+^5ias

where a, i5 > 0 and a -f /3 < L

EXAMPLE 5.2.13. Define F(ii,i2,---.te) :3i+->3fi as

l5 + ^6

The verifications of Examples 5.2.3-5.2.13 are easy, hence details are omitted.

5.3. RESULTS FOR SINGLE-VALUED MAPPINGS

We begin with the following observation.

LEMMA 5.3.1. Let yl,B,5 and r be self mappings of a metric space *(X, d)* satisfying conditions (dis), (die) and

(ei) for *allx^yeX and F eQ*

F{d{Ax,By\d{Sx,Ty),d{Ax,Sx),d{By,Ty),d{Sx,By),d{Ty,Ax)) < 0. (5.1)

Then the pairs *(A,S)* and *{B,T)* share the common property *(E.A).*

PROOF, in view of condition (dis), there exists a sequence *{xn}* in *X* such that

lim *Axn* = lim *Sxn* = t, for some *t E X.*

Since *A{X)* c *T{X),* hence for each *{xn}* there exists {?/„} in *X* such that *Axn* = *Tyn-* Therefore, lim Ty„ = hm *Axn* — *t.* Thus, in all we have *Ax^* -^ *t, Sxn* -^ *t* and *Tyn* -^ *t.* Now, we assert that *Byn* —^ t. If not, then using (5.1), we have

F{d{Axn, Byn), d{Sxn, Tyn), d{Axn, Sxn), d{Byn, Tyn), d{SXn, Byn),d{Tyn, Axn)) < 0

which on making *n* —^ *oo,* reduces to

F{d{t, Byn), 0,0, *d{Byn, t), d{t, Byn),* 0) < 0

a contradiction to (Fj). Hence By„ —> *t* which shows that the pairs *{A,S)* and *{B,T)* satisfy the common property *{E.A).*

REMARK 5.3.1. The converse of Lemma 5.3.1 is not true in general. For a counter example, one can see Example 5.4.1.

THEOREM 5.3.1. Let *A, B, S* and *T* be self mappings of a metric space *{X,d)* satisfying conditions (rfig), (dig) of Theorem 4.6.1 and inequahty (5.1).

Then the pair (^4, *S) as* well as *{B, T)* have a point of coincidence each. Moreover, if the pairs *{A, S)* and *{B, T)* are weakly compatible, then *A, B, S* and *T* have a unique common fixed point in ^.

PROOF, in view of (rfig), there exist two sequences {x„} and {y„} in *X* such that

$$\lim_{n\to\infty} Axn = \lim 5x„ = \text{Um } Byn = \lim Ty„ = t$$

for some *t£X.*

Since 5(X) is a closed subset *oiX,* hence $\text{Um}_{n\to\infty}$ *Sxn = t e S{X).* Therefore, there exists *u e X* such that *Su* = t. Now, we assert that *Au = Su.* If it is not, then *d{Au, Su)* > 0. Using (5.1), we have

F{d{Au, Bt/n), rf(5u, ry„), *d{Au, Su), d{By^,* ry„), *d{Su, Byr,),d{J'ynvAu))* '< V -

which on making n —> oo, reduces to < - ^

F(d(.4w, f), d(5M,*t), d{Au, Su), d{t, t), d{Su, t), d{t, Ai^) <0*

or *F{d{Au, Su),* 0, *d{Au, Su),* 0,0, *d{Su, Au))* < 0 ' • . . i : . _ _ ; ^ "

which contradicts (F2) as *d{Au, Su)* > 0. Hence *Au* = Su which shows that u is a coincidence point of the pair *{A, S).*

Also *T(X)* is a closed subset of *X.* Therefore $\text{Um}_{n\to\infty}$ ry„ = *t* G *T{X).* Hence

Tiu = t for some *w e X.* Suppose *d{Tw, Bw)* > 0, then again using (5.1)

F{d{Axn,Bw),d{Sxn,Tw),d{Axn,Sxn),d{Bw,Tw),d{Sxn, Bw),d{Tw, Ax„)) < 0

which on making n —• 00, reduces to

F{d{t, Bw), d{t, Tw),d{t, t), d{Bw, Tw), d{t, Bw), d{Tw, t)) < 0

or *F{d{Tw, Bw),* 0,0, *d{Bw, Tw),d{Tw, Bw),* 0) < 0

which contradicts (Fi) as *d{Tw, Bw)* > 0. This shows that lu is a coincidence point of the pair *{B,T).*

Since the pairs *{A,S)* and *(B,T)* are weakly compatible, therefore

At = ASu = SAu = St, and *Bt = BTw = TBw = Tt.*

Suppose that *d{At,t)* > 0. Using (5.1), we have

F{d{At, t), d{St, t), d{At, St), d{t, t), d{St, t),d{t, At)) < 0

$$\text{or} \quad F(d(At, t), d(At, t), 0, 0, d(At, t), d(t, At)) < 0$$

which is a contradiction to (F3). Hence $At = t$. This shows that f is a common fixed point of the pair (A, S). Similarly, one can show that Ms a common fixed point of the pair (B,T). Therefore, t *is a.* common fixed point of the mappings A, B, S and T. The uniqueness of the common fixed point is an easy consequence of the condition (F3). This completes the proof.

THEOREM 5.3.2. The conclusions of Theorem 5.3.1 remain true if the condition (dig) of Theorem 5.3.1 is replaced by following.

(62) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

As a corollary of Theorem 5.3.2, we can have the following result which is also a variant of Theorem 5.3.1.

COROLLARY 5.3.1. The conclusions of Theorems 5.3.1 and 5.3.2 remain true if conditions (dig) and (62) are replaced by the following.

(es) $A(X)$ and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

THEOREM 5.3.3. Let A,B,S and T be self mappings of a metric space (X,d) satisfying conditions (^22)) (^23). (^24) and inequality (5.1).

Then the pair (A, S) as well as (B, T) have a point of coincidence each. If the pairs (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

PROOF, in view of Lemma 5.3.1, the pairs (A,S) and (B,T) share the common property $(E.A)$, i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in ^ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $S(X)$ is a closed subset of X, then on the fines of Theorem 5.3.1, the pair (A, S) has a point of coincidence, say u, i.e. $Au = Su$. Since $Au \in A(X)$ and $A(X) \subset T(X)$, there exists $w \in X$ such that $Au = Tw$. Now we assert that $Bw = Tw$. If not, then using (5.1), we have

$$F(d(Ax_n, Bw), d(Sx_n, Tw), d(Ax_n, Sx_n), d(Bw, Tw), d(Sx_n, Bw), d(Tw, Ax_n)) < 0$$

which on making $n \longrightarrow \infty$, reduces to

$$F(d(t, Bw), d(t, Tw), d(t, t), d(Bw, Tw), d(t, Bw), d(Tw, t)) < 0$$

$$\text{or} \qquad F(d(Tw, Bw), 0, 0, d(Bw, Tw), d(Tw, Bw), 0) < 0$$

a contradiction to (F_1). Hence $Bw = Tw$, which shows that tu is a coincidence point of the pair (B, T). Rest of the proof can be completed on the Unes of Theorem 5.3.1. This completes the proof.

By choosing A, B, S and T suitably, one can deduce corollaries for a pair as well as for two different triod of mappings. Now, by setting $B = A$ and $T = 5$ in Theorem 5.3.1, we have the following result for a pair of two self mappings.

COROLLARY 5.3.2. Let A and S be self mappings of a metric space (X,d) satisfying the conditions (das), ((^27) and

(64) for all $x, y \in X$ and $F \in Q$

$$F(d(Ax, Ay), d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), d(Sx, Ay), d(Sy, Ax)) < 0. \tag{5.2}$$

Then A and S have a point of coincidence. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

COROLLARY 5.3.3. Let A, B and 5 be self mappings of a metric space (X,d) such that

(65) the pairs (A, S) and (B, S) enjoy the common property $(E.A)$,

(ee) for all $x, y \in X$ and $F \in Q$

$$F(d(Ax,By),d(Sx,Sy),d(Ax,Sx),d(By,Sy),d(Sx,By),d(Sy,Ax)) < 0, \tag{5.3}$$

(67) $S(X)$ is a closed subset of X.

Then the pair (A, S) as well as $(5, S)$ have a point of coincidence each. Moreover, if the pairs (A, S) and (B, S) are weakly compatible, then A, B and S have a unique common fixed point.

COROLLARY 5.3.4. Let A, S and T be self mappings of a metric space (A',d) such that (e8) the pairs (A,S) and (A,T) share the common property $(E.A)$,

(e9) for all $x, y \in X$ and $F \in \Phi$

$$F(d(Ax,Ay), d(Sx,Ty), d(Ax,Sx), d(Ay,Ty), d(Sx,Ay), d(Ty,Ax)) < 0, \tag{5.4}$$

(e10) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pair (A,S) as well as (A,T) have a point of coincidence each. Moreover, if the pairs (A,S) and (A,T) are weakly compatible, then A, S and T have a unique common fixed point.

COROLLARY 5.3.5. The conclusions of Theorem 5.3.1 remain true if condition (5.1) is replaced by one of the following: (for all $x \neq y \in X$,)

(e11) $d(Ax,By) < \max\left\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{[illegible]}, \frac{d(Sx,By)+d(Ty,Ax)}{[illegible]}\right\}$,

(e12) $d(Ax,By) < \max\{d(Sx,Ty), d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\}$.

(e13) $d^2(Ax,By) < \alpha\max\{d^2(Sx,Ty), d^2(Ax,Sx), d^2(By,Ty)\}$

$$+ \beta\max\{d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\} + \gamma d(Sx,By)d(Ty,Ax),$$

where $\alpha, \beta, \gamma > 0$, $\alpha < 1$ and $\alpha + \gamma < 1$.

(e14) $(1+\alpha d(Sx,Ty))d(Ax,By) < \alpha\max\{d(Ax,Sx)d(By,Ty), d(Sx,By)d(Ty,Ax)\}$

$$+\max\left\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}$$

where $\alpha > 0$.

(e15) $d(Ax,By) < \max\left\{d(Sx,Ty), \frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Sx,By)+d(Ty,Ax)}{2},\right.$

$$\left.\frac{d(Ax,Sx)+d(Sx,By)}{2}, \frac{d(By,Ty)+d(Ty,Ax)}{2}\right\}.$$

(e16) $d(Ax,By) < \alpha[\max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty),$ [illegible]

$$+(1-\alpha)[\max\{d^2(Sx,Ty), d(Ax,Sx)d(By,Ty), d(Sx,By)d(Ty,Ax),$$

$$d(Ax,Sx)d(Ty,Ax), d(By,Ty)d(Sx,By)\}]^{\frac{1}{2}}$$

where a e (0,1) and 0 < /? < 1.

(ei7) *d\Ax, By)* < *ad\Ax, By)d{Sx, Ty)+fid{Ax, By)d{Ax, Sx)d{By, Ty)*

+ 7rf^(5z, *By)d{Ty, Ax)* + *r)d{Sx, By)d\Ty, Ax),*

where a,/3,7,77 > 0 and a + 7 + T/< 1.

(eis) *d{Ax,By)* < ^(-^^»•gx)d(By,Ty) + d(5x,gy)d(Ty, Ax) / 1 + *d{Sx, Ty)*

where Q, /3 > 0 and a + /? < 1.

(e2o) rf(>la;,By) < *<}){mQx{d{Sx,Ty),d{Ax,Sx),d{By,Ty),d{Sx,By),d{Ty,Ax)]),*

where 0 : 3?+ ^ 9? is upper semi-continuous function such that 0(0) = 0 and ^*{t)* < *t* for all *t* > 0.

(621) *d{Ax,By)* < *(l>{d{Sx,Ty),d{Ax,Sx),d(By,Ty),d(Sx,By),diTy,Ax)),*

where 0 : 9?^ -4 3? is upper semi-continuous and nondecreasing function in each coordinate variable such that ^(f,*t,at,bt,ct)* < *t* for each i > 0 and *a,b,c>0* with *a* + *b* + *c<3.*

where a, /? > 0 and a -I- /9 < 1.

rp WMr R,/W^r9rT„>i ^(^=^> 5x)d(gy, Ty) -K rf(5x, *By)d{Ty, Ax)*

(623) rf(Aa:, By) < *d{Sx, Ty)* — *d{Sx,By)+ d{Ty,Ax)* — "

PROOF. The proof follows from Theorem 5.3.1 and Examples 5.2.1-5.2.13.

REMARK 5.3.1. Corollaries corresponding to contraction conditions (cn) to (623) are new results as these never require any condition on containments of ranges of involved mappings. The majority of results corresponding to various above contraction conditions present generalized and improved versions of numerous existing results which include Aamri and Moutawakil [1], Edeistein [28], Chugh and Kumar [21], Fisher [33], Husain and Sehgal [50], Jeong and Rhoades [69], Jungck [71], Kasahara and Rhoades [86], Liu et al. [94], Meade and Singh [96], Pant [102,104], Pant and Pant [109], Park [112-114], Tas et al. [149], Telci et al [150] and some others besides yielding some results which are seeming new to the literature (e.g. 612,618,619,622,623).

COROLLARY 5.3.6. Let J4, JB, *S* and *T* be self mappings of a metric space (X, *d)* which satisfy conditions *(dis)* and (dig) and

d{Ax,By) < 0 (^max|cf(Sx,*Ty),d{Ax,Sx),d{By,Ty),* ^^^'^'^^^ + ^^^^-^""^ |^

where 0 : S?"*" —> 3fi is an upper semi-continuous function such that ^(0) = 0 and *<p{t)* < *t* for all *t* > 0.

Then the pair (A, *S)* as well as *{B, T)* have a point of coincidence each. Moreover, *A, B, S* and T have a unique common fixed point provided the pairs *[A, S)* and (B, *T)* are weakly compatible.

PROOF. Notice that

d{Ax,By) < *cp* *(m^xLsx,Ty),d{Ax,Sx),d{By,Ty),* ^^'^'^'^^^ + ^^^^'^"^^*X)*

< *(t>{max{d{Sx,Ty),d{Ax, Sx),d{By,Ty), d{Sx, By),d{Ty, Ax)}).*

Now proof follows from contractive condition (620) of Corollary 5.3.5.

5.4. ILLUSTRATIVE EXAMPLES

First, we present an example demonstrating the validity of the hypotheses and degree of generaUty of Theorem 5.3.1 over the majority of earlier results proved till date with rare possible exceptions.

EXAMPLE 5.4.1. Consider *X* = (-1,1) equipped with the usual metric. Define self mappings *A, B, S* and 7" on A" as follows:

	(3/5,	if - 1 < x < -1/2		(1/2,	**if - 1 < X < -1/2**
A{x)	f,	if - 1/2 < a: < 1/2	*S{x)* = <	|,	**if - 1/2 < x < 1/2**
	\ ^/5'	if 1/2 < x < 1,		^ ^,	**if 1/2 < X < 1**

' |, if - 1< X < -1/2 r ^, if - 1< X < -1/2

Bix)=l=f; i f - l / 2 < x < l / 2 T (x) = < ^ , i f - l / 2 < x < l / 2

^ |, if l / 2 < x < 1, (|, if l / 2 < x < 1.

Consider the sequences {x„ = ; ^ } and {y„ = ^} in X. Then clearly,

$$\lim_{n\text{-*}oo} Axn = \lim_{n\text{-¥}oo} Sxn = \lim_{n\text{-¥}oo} \mathrm{Bt/n} = \lim_{n\longrightarrow oo} \mathrm{Tyn} = 0$$

which shows that the pairs (.4,5) and (5,T) share the common property *(E.A)*. Also, *A{X)* = *BiX)* = {2} U [-i,|] ?! 5(X) = r(X) = {^,i} U *[^,\]*. Define a continuous implicit function F : 3?^ -> 9? such that *F(ti,t2,..-,tQ)* = *ti* — max{^2, ^^^, ^^*Y^}* where *F e Q*. By a routine calculation, one can verify the inequality (5.1) for *all x ^ y G X*. Thus, all the conditions of Theorem 5.3.1 are satisfied and 0 is a unique common fixed point of the pairs *{A, S)* and *{B,T)* which is their point of coincidence as well.

Here it is worth noting that none of the earlier theorems can be used in the context of this example as Theorem 5.3.1 never require any condition on the containment of ranges of mappings while compactness of the space is replaced by closedness of subspaces. Moreover, the continuity requirements of involved mappings are completely relaxed whereas all earlier theorems (prior to 2001) require the continuity of at least two involved mappings.

Finally, we furnish the following example in support of Theorem 5.3.2 whenever Theorems 5.3.1 and 5.3.3 are not apphcable.

EXAMPLE 5.4,2. Consider *X* = [2,20] equipped with the usual metric. Define self mappings *A,B,S* and T on X as follows:

A2 = 2, Ax = 3, if xe(2,20],

Bx = 2, if X e {2} U [5,20], Bx = 6, if x G (2,5),

52 = 2, *Sx* = ^ ^ , if X e (2,5), *Sx* = 8, if x G [5,20],

T2 = 2, Tx = X -h 8, if X G (2,5), *Tx* = ^ ^ , if x e [5,20].

Clearly, the pairs *{A,S)* and (B,T) satisfy the common property *{E.A).* Also *A{X)* = {2,3} C {2} U [3,13) = *T{X)* and *B{X)* = {2,6} C {2,8} U (5,13/2) = *S{X).* Define a continuous implicit function F as in Example 5.4.1. One can easily verify the inequahty (5.1) *for aU x ^ y e X.* Thus, all the conditions of Theorem 5.3.2 are satisfied and 2 is a imique common fixed point the pairs *{A, S)* and (B,T) which is their coincidence point as well. Here it may be noticed that all the mappings in this example eire still discontinuous at their common fixed point 2.

5.5. **IMPLICIT FUNCTION II**

Recently, Popa [125] utiUzed impUcit functions to prove results for pairs of hybrid mappings which imifies several contraction conditions at the same time and proved an interesting common fixed point theorem. To describe the imphcit function of Popa [125], let A be the set of all real lower semi-continuous functions *F(ti,* ^2, • • •, ^e) : 3?+ -> 3? which satisfy the following conditions *d, G2* and Ga:

(Gi) : *F* is non-increasing in variables ^2, • • • ,*6 and non-decreasing in ti,

(G2) : There exists /i G (0,1) and A; > 1 with *hk <1* such that

$$u < kt \text{ and } F\{t,v,v,u,u + v,0) < 0 \text{ implies } t < hv$$

(G3): *F(t,t,t,o.t,t)* *>oyt>0.*

Popa furnished the following examples of implicit functions in [125].

EXAMPLE 5.5.1. Define F(ti,i2,...,t6) :3ff^-^3fJ as

$$F\{ti,t2,...,t6) = ti-kmaix.<t2,t3,U,-\{t5 + te)> , \quad \text{where fc € } (0,1).$$

EXAMPLE 5.5.2. *De^ne F{ti,t2,...,k): ^l-^ n as*

$$F(ti,t2,...,te) = h\text{-}kl\text{^}amax< t2,ts,U, \text{-}(\text{^}5 + h) \backslash$$

$$\text{-I-} (1 - a) \max \backslash tj,tsU,hte, -tste, -Uh \backslash\backslash \quad J,$$

where A; G (0,1) and 0 < a < 1.

EXAMPLE 5.5.3. $De\&aeF\{ti,t2,...,h)-K-$^^^

$$F\{tut2,...yh) = ti- k[max\{tl,tzU,t^U,UtQ,tzt^\}\backslash 2,$$

where k € (0, -4).

EXAMPLE 5.5.4. Define$F(ti,ti,...,t6)$: J?^ -^»as

$$F\{ti, <2,..., U) = <? + *? + 7 - \wedge - [atl + htl + ctll$$

where $0 <a + b + c<l.$

Implicit functions are quite fruitful in unifying many known results besides deducing the ones not already known. To strengthen this view point we add some more examples to this effect. In this continuation we notice that some other contraction conditions (e.g. rational inequaUties, generalized contraction and several others) can be expressed as implicit functions. Thus we observe that proving fixed point theorems via impHcit functions is a more fruitful idea as a specific impUcit function covers several contraction conditions rather than one contraction condition which is evident from the examples furnished in this section.

EXAMPLE 5.5.5. Define F(ti,t2,...,t6):3fi^-^3(l as

$$F\{ti,t2,... ,te) = h - a \frac{tl + tl}{tz + U} - I3[h + te] \sim 7t2,$$

where 0 < 2a -I- 2^3 -I- 7 < 1.

Gi: Obvious

G2: Let u > 0, u < fct and $F\{t, v, v, u, u+v,$ 0) < 0 where 1 < fc <——————then

————j $-(3\{u+v)-\wedge v$ < 0 which implies t < a (—————1 +$P\{u+v) + 'yv.$ Let on contrary that $u > v$, then $u < k\{a$ + 2/3-1- 7)u < u, a contradiction. Thus $u<v$ $aLndt< k\{a + 2p + 'y)v = hv$ where /i = a -I- 2^ -I- 7 G (0,1) and $hk < I.$ If u = 0 then $t < hv.$

$Gz : F\{t,t,$i,0,$t,t) = t - at - 2/3t - \wedge > 0$ for all t > 0.

EXAMPLE 5,5.6. Define F(ti,t2,...,t6) :3fl^ ^3fl as

$$F\{ti,t2,...,te) = ti -a \frac{'tl + tP}{t\$ + \wedge 6} -P[t3+U]-\wedge t2,$$

where $0 < 2a + 2^\wedge + 7 < 1$.

The verification of requirements *(Gi)*, (G2) **and** *{G3)* is similar, hence it is omitted.

EXAMPLE 5.5.7. *DeGneF(tut2,...,h):^+->Uas*

$$F\{ti, ^2, \bullet\bullet\bullet, te) = <i - a*2 \sim \qquad h + t4$$

where $1 < 2a + c < 2, 0 < 2a + 6 < 2$ and $p > 1$.

Gi : Obvious.

G2 • Let $u > 0, u < kt$ and $F\{t,v,v,u,u + v,0) < 0$ where $\mathbf{1} < fc < \frac{}{2a+c^\wedge}$ then $tP \frac{cwr}{\_u-\backslash-v} av'' < 0$ which impUes $V < \frac{cvw^\wedge}{} + axf$. Let on contrary that $u > V$, then $u < fcP^{'2a+c'} w < u$, a contradiction. Thus $u < v$ and

$t < I \wedge] \; v = hv$ where $h = [\quad)'' \wedge (0,1)$ and /ifc < 1.

G3 : $F(i, t, t, 0, t, 0 = 5(2 - (2a + b))t^\wedge > 0$ for aU <> 0.

5.6. RESULTS FOR PAIRS OF HYBRID MAPPINGS

In this section, we prove a hybrid common fixed point theorem involving a sequence of multi-valued mappings and a pair of self mappings which extends recent common fixed point theorem of Popa [125]. In process, some recent results due to Pathak and Khan [120], Shrivastva et al. [139], Imdad et al. [60] and some others also generalized to a sequence of hybrid pairs of mappings.

The following lemma will be frequently used in the sequel.

LEMMA 5.6.1. **[101]** Let A-B e *CB{X)* and A; > 1. Then for each *aeA*, there exists a point *be B* such that *d{a, b) < kH{A, B)*.

THEOREM 5.6.1. Let {T„}~^i be a sequence of multi-valued mappings of a metric space *{X, d)* to *CB{X)*, whereas /, *J* be self mappings of *X* satisfying

$$F\{H(TiX,Tiy),diIx,Jy),d(TixJx).d\{Tjy,Jy),d\{Ix,Tjy),d\{Jy,Tix)) < 0 \qquad (5.5)$$

where $i = 2n-l, j = 2n\{neN)$ and for all x 7^ y G *X*, with F € A. Suppose that

t^ S^«

(6.2.4) $Ti\{X)$ C $J\{X)$ and $Tj\{X)$ C $I\{X),$

(6.2.5) $I\{X)$ or $J\{X)$ is a complete subspaces of X.

Then the pak (Tj, /) as well as (7), J) has a point of coincidence each.

Moreover, if the pairs $\{Ti, I)$ and (7), $J)$ are quasi-coincidentally commuting and coincidentally idempotent, then $z = IzE\ TiZ$ and $w = Jw\ e\ TjW.$

PROOF. Let $XQEX$ and yi be an arbitrary point in $TIXQ.$ Then there is a xi 6 X such that $Jxi = yi$ which is possible as $Ti\{X)$ C $J\{X).$ By Lemma 5.6.1, one can find a point ^2 € T2a:i such that d(t/i,j/2) < fci/(TiXo,T2Xi) where fc > 1. Let us set y2 = 1^2 as T2(X) C /(X). Thus in general, one can construct a sequence as follows:

2/2n+i = $Jx2n+i$ ^ $TiX2n$ and j/2n+2 = •fa;2,,+2 ^ $TjX2n+i$ such that

$$d(y2n+l, J/2n+2) < kH\{TiX2n,TjX2n+l), \qquad n = 0, 1, 2, . . . \tag{5.6}$$

Now for n > 1, we have

$$F\{H\{TiX2n, TjX2n+l), d\{IX2n, JX2n+l), d\{TiX2n, IX2n), d\{TjX2n+l, JX2n+l),$$

$$d\{IX2n,TjX2n+l),d\{JX2n+uTiX2n)) < 0$$

$$\text{or} \qquad F\{H\{TiX2n, TjX2n+l), d\{y2n, y2n+l), d(y2n+U ?/2n), rf(?/2n+2, y2n+l),$$

$$d(y2n,?/2n+2),4y2n+l,y2n+l)) < 0$$

$$\text{or} \qquad F\{H\{TiX2n, TjX2n+l), d\{y2n, y2n+l), C?(2/2n, y2n+l), rf(y2n+l, ?/2n+2),$$

$$d(j/2n, y2r,+l) + rf(y2n+l, 2/2n+2), 0) < 0. \tag{5.7}$$

By using (5.6), (5.7) and (G2), we have

$$H\{TiX2n, TjX2n+l) < hd\{y2n, y2n+l). \tag{5-8}$$

Now from (5.6) and (5.8), we get

$$\%2n+l, y2n+2) < kh <f(y2n, ?/2n+l)- \tag{5-9}$$

Using similar argument, one can easily show that

$$diy2n,y2n+l) < kh\ d\{y2n-l, y2n) \bullet$$

Since $kh < 1$, therefore the sequence $\{y_n\}$ described by

$$\{Ix_0, Jx_1, Ix_2, Jx_3, \ldots, Ix_{2n}, Jx_{2n+1}, Ix_{2n+2}, \cdots \}$$

is a Cauchy sequence in X.

First, we suppose that $J(X)$ be a complete subspace of X, then note that the subsequence $\{y_{2n+1}\}$ is contained in $J(X)$ and must has a limit $u \in J(X)$. Let $z \in J^{-1}(u)$. Then $Jz = u$, whereas the sequence $\{y_{2n}\}$ being a subsequence of $\{y_n\}$ also converges to u. Now using (5.5)

$$F(H(T_1x_{2n}, T_2z), d(Ix_{2n}, Jz), d(T_1x_{2n}, Ix_{2n}), d(T_2z, Jz),$$

$$d(Ix_{2n}, T_2z), d(Jz, T_1x_{2n})) < 0$$

$$F(d(y_{2n+1}, T_2z), d(y_{2n}, Jz), d(y_{2n+1}, y_{2n}), d(T_2z, Jz), d(y_{2n}, T_2z), d(Jz, y_{2n+1})) < 0.$$

Now making $n \to \infty$, we get

$$F(d(Jz, T_2z), 0, 0, d(T_2z, Jz), d(T_2z, Jz), 0) < 0.$$

Since $d(Jz, T_2z) < kd(Jz, T_2z)$, then in view of (G2), we have $Jz \in T_2z$.

Since $u = Jz \in T_2z$ and $T_2(X) \subset I(X)$ implies that $u \in I(X)$. Let $w \in I^{-1}(u)$. Then $Iw = u$. Suppose $d(T_1w, Iw) > 0$. Then from (5.5),

$$F(H(T_1w, T_2x_{2n+1}), d(Iw, Jx_{2n+1}), d(T_1w, Iw), d(T_2x_{2n+1}, Jx_{2n+1}),$$

$$d(Iw, T_2x_{2n+1}), d(Jx_{2n+1}, T_1w)) < 0$$

$$\text{or} \quad F(d(T_1w, y_{2n+2}), d(Iw, y_{2n+1}), d(T_1w, Iw), d(y_{2n+2}, y_{2n+1}),$$

$$d(Iw, y_{2n+2}), d(y_{2n+1}, T_1w)) < 0$$

which on making $n \to \infty$, reduces to

$$F(d(T_1w, Iw), 0, d(T_1w, Iw), 0, 0, d(T_1w, Iw)) < 0$$

$$\text{or} \quad F(d(T_1w, Iw), d(T_1w, Iw), d(T_1w, Iw), 0, d(T_1w, Iw), d(T_1w, Iw)) < 0$$

a contradiction to (G3). Hence $d(T_1w, Iw) = 0$. Therefore $Iw \in T_1w$.

If one assumes that *I{X)* is complete, then analc^ous arguments establish the earUer conclusions.

Since ^ is a coincidence point of (7}, *J)* therefore using quasi-coincidentally commuting property of *{Tj, J)* and coincidentally idempotent property of J w.r.t *Tj* one can have

$$Jz \text{ G } Ti\{z) \text{ and } u = Jz \wedge Ju = JJz = Jz = u,$$

therefore *u = Ju = JJz* G *JTj{z)* C *Tj{Jz) = Tj{u)* which shows that *u* is the common fixed point of *(Tj, J)*. Similarly using the quasi-coincidentally commuting property of (Tj,/) and coincidentally idempotent property of / w.r.t Tj. One can show that pair (Ti, /) has a common fixed point. This completes the proof.

By setting *Ti = T* (for any odd integer *i e N)* and *Tj = S* (for any even integer *j e N)* in Theorem 5.6.1, we have the following result for two pairs of hybrid mappings.

COROLLARY 5.6.1. Let (X, *d)* be a metric space, *I,J:X-^XandT,S:X^ CB{X)*. Suppose that inequality (5.5) satisfy for /, J, *S* and T,

(626) *T{X)* C *J{X), S{X)* C /(X),

(627) *I{X)* or *J{X)* is a complete subspace of *X.*

Then, the pair (T, /) as well as (5, *J)* have a point of coincidence each.

Moreover, if the pairs (T, /) and *[S, J)* are quasi-coincidentally commuting and coincidentally idempotent, then *z== Iz eTz* and *w = Jw e Sw.*

REMARK 5.6.1. if we set Ti = Tj = r and / = J = F, then we get the improved and generalized form of a result due to Popa [125, Theorem 4].

COROLLARY 5.6.2. The conclusions of Theorem 5.6.1 remain true if for all distinct *x,y in X* impUcit function (5.5) is replaced by one of the following:

(628) *H{TiX,Tjy)* < *kma^l^d{Ix,Jy),diTiX,Ix),diTjy,Jy),^[d{Ix,Tjy)*

+d(Jy,Tix)]Y A; €(0,1).

(ess) *H{TiX,Tjy)* < *k[amsx{d{Ix,Jy),d{TixJx),d{Tjy,Jy),-[d{Ix,Tjy)*

+d{Jy, Tix)] + (1 - *a)[mBx{d\lx, Jy), d{TiX, Ix)d{Tjy, Jy),*

d(Ix, Tjy)d{Jy, Tix), ^d(TiX, Ix)d{Jy, Tix), ^diTjy, Jy)d{Ix, T,y)}] 5],

where *k e* (0,1) and 0 < a < 1.

(eso) *H{TiX,Tjy)* < *k[max{d:^{Ix,Jy),d{TiX,Ix)d{Tjy,Jy),d{Ix,Tjy)d{Jy,Tix),*

d{Tjy, Jy)d{Ix, Tjy), d{TiX, Ix)d{Ix, Tjy)}] ^,

where *k e {0,-j=).*

(en) *H\TiX,Tjy)+H\TiX,Tjy)* < *ad'^{Ix,Jy)+bd\TixJx)+cd{Tjy,Jy)*

HjTjX, Tjy)
l+d{Ix,Tjy)d{Jy,Tixy

where 0 < o + 6 + c < l.

-*(P{TiX,Ix)* + *d'{Tjy,Jy)*
(e33)//(T.x,T,,)<. *,^T^,^,^^^,^T,y,Jy)* + *P[d{Ix,Tjy)* + *d{Jy,Tix)]*

+*^d{Ix,Jy),*

where 0 < 2a + 2^ + 7 < 1.

(633) *HiTx T-v)* < *a* *\^iI^^T,y) + <P{Jy,T,x)* *+0[TiX, Ix)+d{Tjy, Jy)\+^d{Ix, Jy),*

where 0 < 2a + 2/? + 7 < 1.

where l < 2 a + c < 2 , 0 < 2 a + 6 < 2 and p > 1.

PROOF. The proof of the Corollary 5.6.2 follows from Theorem 5.6.1 and Examples 5.6.1-5.6.7.

REMARK 5.6.2. Theorem 3 of Pathak and Khan [120] proved for a pair of hybrid mappings is a consequence of Theorem 5.6.1. Also Corollary 5.6.2 corresponding to contraction condition (629) is an extension of Theorem 3 of Pathak and Khan [120] to a sequence of multi-valued mappings.

REMARK 5.6.3. Theorem 3.2 of Imdad et al. [60] proved for pairs of hybrid mappings is also the consequence of Theorem 5.6.1. Whereas Corollary 5.6.2 corresponding to contraction condition (632) is an extension of Theorem 3.2 of Imdad et al. [60] to a sequence of multi-valued mappings.

REMARK 5.6.4. Theorem 2.1 of Shrivastva et al. [139] follows from Corollary 5.6.1. Also Corollary 5.6.2 corresponding to contraction condition (634) is an extension of Theorem 2.1 of Shrivastva et al. [139] to a sequence of multi-valued mappings.

REMARK 5.6.5. Corollary 5.6.2 corresponding to contraction condition *{e2&)* generalizes several known results whereas contraction conditions (630) and (633) are seeming new to the literature.

REMARK 5.6.6. if we take {TJ^i to be a sequence of single-valued mappings and modify rest of the hypotheses of Theorem 5.6.1 accordingly, then we get improved version of Theorem 3 of Sessa et al. [137].

The following corollary can be viewed as single-valued analogue of Corollary 5.6.1.

COROLLARY 5.6.3. Let $\{X,d)$ be a metric space and let S,T,I and J be self mappings of X such that $\{S, I)$ and $\{T, J)$ are weakly compatible with $S\{X)$ C $J\{X)$ and $T\{X)$ c $I\{X)$ satisfying

$$F\{d\{Sx,Ty),d\{Ix,Jy),diIx,Sx),d\{Jy,Tyld\{Ix,Ty),d\{Jy,Sx)) \quad < \quad 0.$$

If one of $S\{X),T\{X), I\{X)$ and $J\{X)$ is complete subspace of X, then there exists a unique common fixed point of S, T, I and J.

PROOF. The proof follows from Theorem 2.1 of Imdad et al. [63].

Chapter 6

RECIPROCAL CONTINUITY AND COMMON FIXED POINTS OF NONSELF MAPPINGS

6.1. INTRODUCTION

In practice, there do exist many situations which cannot be described by self mappings and hence a systematic study of nonself mappings is worth pursuing. Keeping in view this observation, Assad and Kirk [5, 1972] initiated the study of fixed point theorems for nonself mappings in metrically convex metric spaces. In recent years, inspired by Rhoades [127], a multitude of fixed point results for single-valued mappings have been proved by various researchers of this domain. To mention a few, one may cite Assad [6,7], Imdad and Kumar [62], Khan et al. [91]. In 1976, Assad [6] proved some results for nonself mappings defined on a closed subset of a complete metrically convex metric space satisfying Kannan type mappings which have subsequentally been generalized by Khan et al. [91] for generalized type contractions. Recently, Imdad and Khan [54,55] and Imdad et al. [56] generalized these results for pairs of mappings. Here one may note that in such results, one often requires the continuity of all or some of the involved mappings (e.g. [54-56,62,91]).

On the other hand, Park [115] and Khan et al. [90] used a new technique to prove fixed point theorems in metric spaces by altering distances between the points employing suitably equipped with continuous control functions which have further been pursued by Pathak and Sharma [121], Sastry and Babu [131] and Sastry et al. [132]. Assad [6,7], Abdalla and Zaheer [8], Imdad and Khan [54] and others utilized this technique in nonself setting.

The control function employed in Sastry et al. [131] to alter distances is indeed a function 0 :3?^ ->• S?*' which satisfy the following properties:

(/i) 0 is continuous at origin and monotonically increasing in Sf*',

(/2) *<P{t)* ***^0^t = 0,***

ifs) 4>m < 2(t>{t).

REMARK 6.1.1. Let {<„} C 3?+ satisfying ^(f„) -^ 0 as n -^ oo. Then f„ -> 0 as n —>^ CX3.

Before proving our results, we collect the relevant definitions and results.

DEFINITION 6.1.1. Let K be a nonempty subset of a metric space *{X,d)* and *G,S : K* -^ *X.* The pair (G, *S) is* said to be weakly commuting if

d{GSx,SGx)<d{Gx,Sx)

for every *x G K* with *Gx, Sx* 6 *K.*

Note that for *K* = *X,* this definition reduces to that of Sessa [136].

Motivated from [45], we define the (^compatibility for nonself mappings as follows:

DEFINITION 6.1.2. Let /f be a nonempty subset of a metric space *{X,d)* and *G,S : K* —^ *X.* The pair (G, *S)* is said to be (^compatible if

lim *(j){d{GSxr„SGxn))=0*

whenever there is a sequence *{xA* C *K* such that Um *d(Gxn,Sxn)* = 0 with
n-foo
GXn, SXn e K.

Note that *iox K* = *X* this definition reduces to (/^compatibility due to Sastry et al. [132] and for *<j)* = *Id.* and *K* = *X* this definition reduces to compatibihty for self mappings due to Jungck [72].

In view of Remark 6.1.1, ^compatibility and compatibility are equivalent. Hence throughout this chapter, we use compatibihty instead of (^compatibihty.

DEFINITION 6.1.3. A pair *{G,S)* of nonself mappings defined on a nonempty subset *K oi a.* set *X* is said to be weakly compatible if *Gx* = *Sx* for some *x E K* with *Gx, SxeK* => *GSx* = *SGx.*

DEFINITION 6.1.4.[91] Let (x,d) be a metric space and *K he &* nonempty subset of *X.* Let *F,G,S,T : K* ^ *X* which satisfy the inequahty

<Pid{Fx,Gy))<a max!.^(f>{d{Tx,Sy)),<l>id{Tx,Fx)),<f>{d{Sy,Gy))

$$+ 6 \, [4,\{d\{Tx, Gy)) + <\{>\{diSy, Fx))] \tag{6.1}$$

for all distinct $x,y \in K$ with $a,6 > 0$ such that $a + 26 < 1$, where $</\cdot: K^{*'} \to 3?^{*'}$ be a control function which satisfies (/i), (/a) and (/a). Then (F, G) is called generalized $\{T, S)$ contraction mappings of K into X.

The purpose of this chapter is to extend the notions of reciprocal continuity, ^compatibility and C,-commutativity for nonself mappings and use them to prove some coincidence and common fixed point theorems in metrically convex metric spax:es. Our main theorem generalizes earher results due to Assad [6], Imdad and Khan [54], Imdad et al. [56], Khan and Bharadwaj [88], Khan et al. [91] and others. As an application of our main result, we also prove a common fixed point theorem in Banach spaces besides furnishing some illustrative examples.

Thus the first section concludes which has provided an introduction to the contents of this chapter and also incorporates the relevant definitions and results. In Section 6.2, we define the notion of reciprocal continuity for nonself mappings and also prove a lemma. In Section 6.3, we prove some common fixed point theorems for two pairs of nonself mappings besides furnishing an illustrative example. In the last section, we define C,-commutativity for nonself mappings and discuss the equivalence of certain kinds of commutativity. We also prove a common fixed point theorem in Banach spaces.

6.2. RECIPROCAL CONTINUITY IN NONSELF SETTING

Recently, Pant [105] introduced the notion of reciprocal continuity and used it to prove common fixed point theorems for contraction type self mappings. In what follows, we extend this notion to nonself setting and also furnish an example besides proving a related result.

A natural extension of reciprocal continuity to nonself mappings can be given as follows:

DEFINITION 6.2.1. Let K he a nonempty subset of a metric space (X, d). A pair of mappings $G,S : K \to X$ is said to be reciprocally continuous if $\lim GSx_n = Gz$

and $\lim_{n\to\infty} SGx_n = Sz$ **whenever there** is a sequence $\{x_n\} \subset K$ with $\{Sx_n\}, \{Gx_n\} \subset K$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Gx_n = z \quad \text{for some } z \in K.$$

Notice that if $K = X$ this definition reduces to the definition of reciprocal continuity due to Pant [105] given for self mappings. However in the context of above definition the following facts are worth noticing.

(i) If both the component maps G and S are continuous, then they are obviously reciprocally continuous but the converse is not true (see Example 6.2.1 below).

(ii) There exists a rich class of pairs of discontinuous mappings which are compatible as well as reciprocally continuous and this is why the notion of reciprocal continuity is advantageous in proving results on common fixed points. To substantiate our claim, we furnish the following example.

EXAMPLE 6.2.1. Consider $X = [1,\infty)$ equipped with the Euclidean metric d and $K = [1,3]$. Define $G,S : K \to X$ as

$$Gx = \begin{cases} 2x^4 - 1, & 1 < x < 25 \\ x^{?}, & \text{otherwise,} \end{cases} \qquad Sx = \begin{cases} x^2, & 1 < x < 25 \\ 2x^{?} - 1, & \text{otherwise.} \end{cases}$$

Then for any sequence $\{x_n\} \subset K$ with $\{Gx_n\}, \{Sx_n\} \subset K$, we have

$$d(GSx_n, SGx_n) = |2x^{?} - x^{?} - 1| \to 0 \Leftrightarrow x_n \to 1 \Leftrightarrow Gx_n \to 1 \text{ and } Sx_n \to 1.$$

Thus the pair (G, S) is compatible on K. Also notice that for sequence $\{x_n\}$ with $Gx_n \to 1$ and $Sx_n \to 1$, we have $\lim_{n\to\infty} GSx_n = 1 = G1$ and $\lim_{n\to\infty} SGx_n = 1 = S1$ which shows that the pair (G,S) is reciprocally continuous whereas both the components mappings are discontinuous.

Here, we point out that common fixed point theorems for compatible mappings often require the continuity of some or all involved mappings (e.g. [105, Theorem 3.2]). The notion of reciprocal continuity makes it possible to prove common fixed point theorems for compatible mappings under relatively less continuity requirement. In the setting of common fixed point theorems, pair of compatible mappings satisfying a suitable contractive or contraction condition can ensure reciprocal continuity in the presence of continuity of one of the mappings. In the sequel, we prove the following lenuna which exhibits that under a contraction condition patterned after Khan et al. [105] in a nonself setting in metrically convex metric spaces compatibility of paurs impUes their reciprocal continuity provided one component map of a pair is continuous.

LEMMA 6.2.1. Let A" be a subset of a metric space *(X,d)* and let *F,G,S,T* : *K* ^ *X he* four mappings such that *{F,G)* be a generalized *(T,S)* contraction mappings of *K* into *X.* Suppose that

(/e) *GKnKcTK* (resp. *FKnKc SK),*

(fj) the pair *{G,S)* (resp. *{F,T)) is* compatible,

(/g) G or 5 (resp. F or T) is continuous,

(/g) the control function </>: 3?+ -4 3?+ is left continuous.

Then the pair *{G,S)* (resp. (F,T)) is reciprocally continuous.

PROOF. Let us assume that the pair *{G,S)* is compatible and *S* is continuous with *{xn}* in *K* such that Gx„ -4 *z,* 5x„ -^ *z* and *{Gxn},* {5x„} C *K.* Since *S* is continuous, we get 5Gz„ -4 *Sz* and 55x„ -4 *Sz.* Now compatibiUty of (G, 5) implies that Um d(G5x„, 5Ga;„) = 0, that is *GSXn* ->• *Sz.* Since *GKnK CTK* for
n—>oo

each x„ there exists some t/„ in T/f with GS'Zn = ry„. Then SSxn ->• *Sz,* 5Gx„ -> 52, GS'xn -^ *Sz* and rj/„ -> *Sz.* We assert that Fy„ -> Sz. If not, then there exists a subsequence *Fym,* a number e > 0 and a positive integer *UQ e N* such that for each m > no, we have *d{GSxm, Fym)* > e, *d(Fymi Sz)* > *e.* Now using (6.1), we have

4>{d{Fym, GSxm)) < a max { *^<p{d{Tym, SSxm)), <l>{d{Tym, Fym)), <l>{diSSxm, GSxn*

+b[<f>{d{Tym, GSxm)) + *(f>{d{SSxm, Fym))*

which on letting m —> oo, reduces to

<l>{d{Fym, Sz)) < (a + *b)<f>{d{Fym, Sz))* < *<f>{d{Fym, Sz))*

which is a contradiction. Hence lim Fy„ = *Sz.* If *Gz* ^ *Sz,* then by (6.1), we get
n—¥oo

(f>id{Gz, Fy^)) < a max I ^0(rf(ry„, 5^)). *4>id{Tyn, Fyn)), <f>(d{Sz, Gz))*

+ 6[<^(rf(ry„, *Gz))* + *<j>{d{Sz, Fyn))]*

which on letting n -f oo, reduces to

<f>{d{Gz, Sz)) <{a + b)<f>{d{Sz, Gz))

< *<t>{d{Sz,Gz))*

a contradiction. Hence *Sz* = *Gz.* Thus by assuming 5x„ -^ *z* and Gx„ -> 2; along with continuity of 5, we obtain 5Gx„ -> ^z and G5a;„ -4 *Gz{= Sz)* which shows that the pair (G, 5) is reciprocally continuous. We arrive at the same conclusion when the pair (G, *S)* is compatible and G is continuous. The proof for the other pair (F, *T)* is similar, hence it is omitted. This completes the proof.

6.3. MAIN RESULTS

Our main result runs as follows:

THEOREM 6.3.1. Let K be a nonempty closed subset of a complete metrically convex metric space *X.* If (F, G) is a generalized (T, *S)* contraction mappings of *K* into *X* which satisfy

(/lo) *dKcSKnTK, FKnKcSK, GKnKc TK,*

ifn) TxedK^Fxe K, Sx e dK ^ Gx e K,

if 12) either the pair (G, *S)* is compatible and reciprocally continuous or the pair (F, *T)* is compatible and reciprocally continuous.

Then the pair (G, *S)* as well as (F, *T)* has a point of coincidence.

Moreover, G, *S, F* and *T* have a unique common fixed point provided the pairs (G, *S)* and (F, *T)* are weakly compatible.

PROOF. Firstly, we proceed to construct two sequences *{xn}* and {y„} in the following way.

Let *X* € *dK.* Since *dK* C *TK,* there exists a point *XQ e K* such that *x* = *TXQ.* Prom the impUcation *TXQ edK => FXQ e KnFK* C *SK.* Let xi G /f be such that yi = 5xi = *FXQ* G A". Since j/i = Fxo> then there exists a point ya = *Gxi* such that

rf(yi.y2) = rf(^a;o,Gxi).

Suppose 7/2 ^ -^^ Then y2 ^ -^ H *GK* C T/T which imphes that there exists a point *X2 G K* such that y2 = *Tx2.* Otherwise, if y2 ^ -ft^, then there exists a point *p* G *dK* such that

d{Sxi,p) + *d{p, Vi)* = *d{Sxi,y2).*

Since *p* G 5A' C TiiT, there exists a point X2 G AT such that *p* = *Tx2* and so

d{SxuTx2) + *d{Tx2,y2)* = rf(5xi, ya)-

Let *1/3 — Fx2* be such that

d{y2,yz) = *d{Gxi,Fx2).*

Thus repeating the forgoing arguments one obtains two sequences {x„} and {y„} such that

(/13) y2n = *GX2n-U* *y2n+l* = *FX2n,*

ifu) y2n e A => y2„ = rX2„ or *y2n^K=^ TX2n* £ *dK*

d{SX2n-i,TX2n) + *d{TX2n,y2n)* = d(5x2„-l, y2n),

(/15) y27i+i *e K* =^ y2„+i = 5x2„+i or y2„+i ^ *K* =^ 5x2„+i G *dK*

d{Tx2n, SX2n+l) + rf('S'X2„+l, y2n+l) = *d{TX2n,y2n+l)*-

We denote

Po = *{Tx2i* G {rx2„} : Tx2i = y2i},

Pi = *{Tx2i* G {rx2„} : rx2i ^ y2i},

Qo = {'S'x2i+i G {5'x2„+i} : 5'x2i+i = y2i+i},

Ql = {'5X2,-1-1 *G* {'S'a;2n+l} : *Sx2i+l* ^ *y2i+l}*-

Note that *{Tx2n,Sx2n+i)* ^ *Pi x Qv* Similarly, (5x2n-i,rx2„) ^ Qi x *Pi.*

Now, we distinguish the foUowing three cases:

CASE 1. if (rx2n,5x2,,+i) *ePox QQ,* then

(l>{d{Tx2n,Sx2n+l)) = <f>{d{FX2n, Gx^n-l))

< *amax.<* *-(f>{d{TX2n,SX2n-l)),<l>{.d{TX2n,FX2n)),<f>{d{SX2n-l,Gx2n-l))*

+b[(f>{d{TX2n. Gx2n-l)) + <f>{d{FX2n, SX2n-l))]

= amax{(^(d(y2n,2/2n-i)),<^(%2n,2/2n+i))} + *b(f){d{y2n-i,y2n+i))*

= amax{0(d(y2n,y2n-l)),<^(d(y2n,y2n+l))} + *b[(f>{d{y2n-l,y2n) + d{y2n,y2n+l))]-*

If *d{y2n,y2n-i) > d{y2n,y2n+i),* then

(f>{d{TX2n,Sx2n+l)) < {a + 2b)(f>{d{TX2n, Sx2n-l)).

Otherwise, if d(y2,,,y2n-i) < *d{y2n,y2n+i),* then we have

(f){d{TX2n,Sx2n+l)) < a(^(d(y2,,, y2n+l)) + 26(/)(d(y2n, y2n+l))

= (a + *2b)(f){d{y2n,y2n+i))*

< *(f>{d{y2n,y2n+l))*

which is a contradiction. Hence

(f>{d{Tx2n, SX2n+l)) < (a + *2b)(j>{d{TX2n, Sx2n-l))-*

Similarly, if *{Sx2n-i,Tx2n)* € Qo x -Po, then

<f>{d{SX2n-.l, Tx2n)) < (o + *2b)<f>(d{SX2n-uTX2n-2))-*

CASE 2. if *{Tx2n,Sx2n+i)* G Po X Qi, then we have

d{TX2n, Sx2n+l) + d{Sx2n+U y2n+l) = d{Tx2n, y2n+l)

which in turn yields

d{TX2n, Sx2n+l) < d{Tx2n,y2n+l) = d{y2n,y2n+l), and hence

$$(f>id\{Tx2n,SX2n+l)) \quad < \quad <f>(d\{TX2n,y2n+l)) \quad = \quad 0(rf(l/2n,y2n+l)).$$

Now, as in Case 1, we obtain

$$(f>\{d\{TX2n,SX2n+l)) \quad < \quad \{a \ +2b)<f>\{d\{TX2n, \ SX2n-l))-$$

In case $\{Sx2n-i,Tx2n) \wedge Qi \ x$ -Fb, then

$$(i>id\{SX2n-uTX2n)) \quad < \quad \{a+ \quad 2h)(l>\{d\{SX2n-l,Tx2n-2))-$$

CASE 3. if d(rx2,,, 5x2,,+i) G PI X QO, then 5x2n-i G Qo and

$$d(Ta;2,,,5a;2,,+i) = d(Tx2,,,y2n+i) < d\{Tx2n,y2n) + rf(y2n,y2n+i)-$$

Note that $d\{y2n, Sx2n+i) = d\{Fx2n,Gx2n-i)$, therefore proceeding as in Case 1, we have

$$0(d(y2n,y2n+l)) \quad = \quad <t>\{d\{FX2n, \ GX2n-l)) \quad < \quad (a \ + \ 2b)(f>\{d\{TX2n, \ Sx2n-l))$$

$$<<f>\{d\{Tx2n,SX2n-l))$$

and thus $d\{y2n,y2n+i) < d\{Tx2n,Sx2n--i)$, as 0 is an increasing function, therefore, we can write

$$d\{TX2n, \ SX2n+l) \quad < \quad d(TX2,,,y2n) \ + \ d\{Tx2n. \ Sx2n-l) \quad = \quad d\{SX2 \ n—Ij \ y2n)$$

and hence

$$(/>(d(ra;2,,,5x2n+l)) \quad < \quad <j>\{d\{Sx2n-Uy2n)) \quad < \quad \{a+ \ 2b)(j)\{d\{Tx2n-2, \ SX2n-l))$$

$$= k<f)\{d\{Tx2n-2, Sx2n-i)), \quad \text{where} \ k = a + 2b.$$

Thus in all the cases, we have

$$(f>\{d\{TX2n, \quad SX2n+l)) \quad < \quad kmaiX.\{(l>\{d\{SX2n-\backslash,Tx2n)), \quad (t>\{d\{TX2n-2, \quad Sx2n-l))\}$$

whereas

$$(f)\{d\{Sx2n+l,TX2n+2)) \quad < \quad k \quad max\{()\}\{d\{SX2n-l,Tx2n)), \quad <P\{d\{Tx2n, \quad SX2n+l))\}-$$

It can be **shown by induction that forn** > **1,**

<f>{d{Tx2n,Sx2a+x)) < k"^-^mBx{<f>{d{Txo,Sx^)),<l>{d{Sxx,Tx2))}

and

<?i(d(5x2n+i,Tx2n+2)) < *k^m.Bx{(i>{d{Sxx,Tx2))A{d{Tx2,Sx^))}.*

Now, for any positive integer p, we have

<P{d{TX2n,SX2n+p)) < 4>{d{TX2n,SX2n^l)'\-d{Sx2n^-uTX2n+2)+- • .+d{Tx2n+p-U Sx2n+p))

< *<f>{{l + k + k^ + ...* + fc"-')A:2"*max{d(TxQ,Sxi),d{SxuTx2)})*

= *<t> ((^^) ^3^{d{Txo,Sxx),d{Sxi,Tx2)}*

which shows that the sequence *{TXQ, Sxi, Tx2,.* • •, Sx2n-i > *Tx2n, Sx2n+i,.* • •} is Cauchy in *K*. Then as noted in [45], there exists at least one subsequence *{Tx2nt}* or *{Sx2n^+i}* which is contained in *PQ* and *QQ* respectively and converges to some *z* in *K as K is a.* closed subset of *X.*

Since the pair *(G, S)* is reciprocally continuous as well as compatible, therefore *{Tx2n^ = Gx2n^-i* and *Sx2nk-i e K)* 5Gx2nfc-i ->• *Sz* and *GSx2n^-i* -> *Gz* and *Gz = Sz.*

Since *GK OK C TK,* there exists a point *w in K* such that *Gz = Tw = Sz.* Now we show that *Gz = Fw.* Suppose that *Gz ^ Fw.* Using (6.1), we have

(i>{d{Fw, Gz)) < (a + *b)(l){d{Fw, Gz)) < 4>{d{Fw, Gz))*

which is a contradiction and hence *Gz = Sz = Fw = Tw.* Thus both the pairs have points of coincidence.

Since the pair (G, 5) is weakly compatible, we have

$$GGz = GSz = SGz = SSz.$$

We show that *GGz = Gz.* Suppose that it is not so, then again using (6.1), we obtain

(t>{d{Gz,GGz)) = (f>{d{Fw,GGz))

$$< (o + \%)(t>\{d\{Gz,GGz)) < 4>\{d\{Gz,GGz))$$

a contradiction and hence $Gz = GGz$.

Also $GGz = G\{Sz) = SGz$, therefore $Gz'vs\ a$, common fixed point of the pair (G,S).

Also, suppose that Fw ^ FFw, then again as above

$$(i>id\{Fw,FFw)) = (f>(d(FFw,Gz))$$

$$<\{a + 2b)(f>\{d\{Fw, FFw)) < <f>\{d\{Fw, FFw))$$

which is a contradiction and hence $FFw = Fw$ and $FFw = FTw = TFw$. This shows that Fw is a common fixed point of the pair $\{F,T)$. Hence Gz is a unique common fixed pomt of G, 5, F and T. The uniqueness of Gz easily follows from (6.1).

If we assume that $\{F, T)$ is compatible pair of reciprocally continuous mappings, then proceeding on similar lines one can estabhsh the earlier conclusions. This completes the proof.

REMARK 6.3.1. Theorem 6.3.1 generalizes earlier results due to Assad [7], Imdad and Khan [54], Imdad et al. [56], Khan and Bharadwaj [88], Khan et al. [91] and others as we never require continuity of the involved mappings.

Before proving further results, we can deduce several results in the form of corollaries for a pair as well as for triod of mappings by choosing G, S, F and T suitably. Here we state only one result for a pair of mappings.

COROLLARY 6.3.1. Let *Khea* nonempty closed subset of a complete metrically convex metric space X and $G,S : K$ -^ X which satisfy inequality (6.1) (for $F = G$ and $T = S$). Suppose that

(/le) $dK\ CSK,\ GKDKC\ SK,$

(/ir) $SxedK$^$Gxe\quad K,$

(/is) the pair (G, 5) is compatible and reciprocally continuous.

Then the pair (G, 5) has a point of coincidence. Moreover, *G* and *S* have a unique common fixed point provided the pair *(G, S)* is weakly compatible.

THEOREM 6.3.2. Let X be a nonempty closed subset of a complete metrically convex metric space *X*. If (F, *G)* be a generalized (T, *S)* contraction mappings of *K* into *X* which satisfy conditions *(fio),* (/ii),

(/19) either pan: *{G,S)* is compatible and *G* (or *S)* continuous or pair *{F,T)* is compatible and *F* (or *T)* continuous,

(/20) the control function (^ : S?*' -> 3?*' is left continuous.

Then the pair (G, *S)* as well as (F, *T)* has a point of coincidence.

PROOF. Suppose the pair *{G, S)* is compatible and *G* continuous, then in view of Lemma 6.2.1, the pair (G, *S)* is reciprocally continuous. Similarly, if we assume compatibility of pau: (G, *S)* and continuity of 5, then the pau: (G, *S)* again reciprocally continuous. Hence the proof follows from Theorem 6.3.1.

Finally, we prove a result when 'closedness of *K'* is replaced by 'compactness of *K*

THEOREM 6.3.3. Let *(X,d)* be a complete metrically convex metric space and *K hea,* nonempty compact subset *of X. Let F,G,T : K* -^ *X* satisfying

(/21) *dK c TK, (FK* U *GK) nKCTK,*

if22) TxedK=^Fx,GxeK,

if23) 4>[d{Fx,Gy)) < M{x,y) with *M{x,y) > 0 iox x,y E K* where

M(x, y) = a max I *^mTx, Ty)), 4>{d{Tx, Fx)), (f>{d{Ty, Gy)) *

+ *b [<f>{d{Tx, Gy)) + <j>{d{Ty, Fx))].* (6.2)

Then F, G and T have points of coincidence provided the pair (F, T) or (G, *T)* is compatible and reciprocally continuous.

PROOF. We assert that M{x,y) = 0 for some x,y e K. Otherwise M{x,y) ^ 0, for any x,y e K. Define

<f>{d{Fx,Gy))
^(^'^^= M{x,y) •

Then f{x, y) is continuous and satisfies /(x, y) < 1 for all (a;, j/) € ii" x /f. Since K X K \s compact, there exists {u,v) G K x K such that /(a;,y) < f{u,v) = c < 1 ioT x,y e K which in turn yields (l>{d{Fx,Gy)) < c M{x,y) for x,y e K and 0 < c < 1. Therefore using (6.2), one obtains ca + 2cb < \. Now by Theorem 6.3.1 (with restriction S = T), one gets Tz = Fz and Tw = Gw for some z,w e K. Consequently M(z, w) = 0, contradicting the fact M{x,y) > 0. Therefore M{x, y) = 0 for some x,y e K which imphes Tx = Fx and Tx = Ty = Gy. If M(x,x) = 0 then Tx = Gx and if M(x, x) ^ 0 then using (6.2), one infers that d{Tx, Gx) < 0 yielding thereby Tx = Gx. Similarly, in either of the cases M{y, y) = 0 or M{y, y) > 0, one concludes that Ty = Fy. Thus we have shown that F, G and T have a common point of coincidence. For fixed point, the proof is identical to that of Theorem 6.3.1, and hence omitted. This completes the proof.

Now, we furnish an example which demonstrates the validity of the hypotheses of Theorem 6.3.1 besides establishing the genuineness of oiu: extension over several other relevant results of the existing hterature.

EXAMPLE 6.3.1. Let X = 3? with the usual metric and K = [0,3]. Define 4>:U+ ^^+ as <p{t) = t&ndF,G,S,T:K-^X as

$$Fx = \begin{cases} x^2, & \text{if } 0 < X < 2 \\ \frac{1}{2}, & \text{if } 2 < x \leq 3, \end{cases} \qquad Tx = \begin{cases} 2x^4, & \text{if } 0 \leq x \leq 2 \\ 3, & \text{if } 2 < x \leq 3, \end{cases}$$

$$Gx = \begin{cases} x^3, & \text{if } 0 \leq x \leq 2 \\ \frac{1}{2}, & \text{if } 2 < x \leq 3 \end{cases} \quad \text{and} \quad Sx = \begin{cases} 2x^6, & \text{if } 0 \leq x \leq 2 \\ 3, & \text{if } 2 < x \leq 3. \end{cases}$$

Since dK = {0,3} and TKnSK = [0,32] n [0,128] = [0,32], hence dK = {0,3} cTKn SK. Further, FK n K = [0,4] n [0,3] = [0,3] C SK and GKHK = [0,8]n[0,3]cTK Also

TO = OEdK=>FO = OeK, T3 = 3 G dK^ FS^-eK, Moreover, if for *x e* [0,2] and y € (2,3], then

d{Fx,Gy) = \x^-'^\

< I *max{ld{Tx,Sy),d{Tx,Fx),d{Sy,Gy)}*

\+ *^diFx,Sy)* + *d{Tx,Gy)}.*

Next, if *x,ye* (2,3], then

d{Fx,Gy) = *0=^d{Tx,Sy).*

< I max{|d(Tx, *Sy), d{Tx,* Fx), *{Sy, Gy)}*

\+ *'^{d{Fx,Sy)* + *diTx,Gy)}.*

Finally, if x,y e [0,2], then

d{Fx,Gy) = \x-'-y^\

< I *m8ix{'^d{Tx,Sy),d{Tx,Fx),{Sy,Gy)}*

\+ *^d{Fx,Sy)* + *d{Tx,Gy)}*

which shows that the contraction condition (6.1) is satisfied for every distinct *x,y* £ *K.*

Notice that the pah (G,S) (also *{F,T))* is compatible and reciprocally continuous (e.g. *Xn* = ^*).* Moreover, 0 is a point of common coincidence as TO = FO and *SO* = *GO* with TFO = 0 = *FTO* and *SGO* = 0 = *GSO* which shows that the pairs *{F,T)* and *{G,S)* are weakly compatible. Thus all the conditions of Theorem 6.3.1 are satisfied and '0' is the unique common fixed point of F, *G, S* and *T.*

Here, it may be pointed out that all the four involved mappings are discontinuous which establishes the utility of our results over the ones hypothesizing continuity requirement.

6.4. C,-COMMUTATIVITY WITH AN APPLICATION

Recently, Al-Thagafi and Shahzad [3] introduced the concept of C,-commuting mappings and proved some common fixed point theorems along with related results on invariant approximations. We extend the notion of C,-commutativity to a pair of nonself mappings. As an apphcation of Theorem 6.3.1, we prove a common fixed point theorem for the Cg-commuting mappings in Bamach spaces.

DEFINITION 6.4.1. Let *{G,S)* be a pair of nonself mappings defined on *q*-starshaped subset *K* of a normed space *X* with *q* G *F(S)* and Cq(G, 5') := *U{C{Gk, S)* 0 < fc < 1} where *GkX* = *kGx* + *{l- k)q*. Then the pair (G, *S)* is said to be *Cg*-commuting if *GSx* = *SGx* for all *x* e *Cq{G, S)* provided *Gx, Sx* e *K*.

Clearly, Cg-commuting mappings are weakly compatible but converse need not be true in general. The following simple example illustrates the situation better.

EXAMPLE 6.4.1. Let X = K with usual norm and *K* = [0,3]. Define *G,S :* *K* ^ *Xhy Gx* = *x*^ for all x 7^ 2 and G2 = 1, and *Sx* = *2x* for all *x* £ *K*. Then *K* is g-starshaped with 9 = 0 G *F{S), C{G, S)* = {0} and Cg(G, *S)* = *K\{2}*. It is easy to verify that *{G, S) is a* pair of weakly compatible nonself mappings but not Cq-commuting.

In an attempt to generalize the notion of weak commutativity due to Sessa [136], Pant [102] introduced the notions of pointwise i2-weak commutativity and /?-weak commutativity for self mappings. Imdad and Kumar [62] extended these notions for a pair of nonself mappings and proved some coincidence and fixed point theorems in metrically convex metric spaces.

DEFINITION 6.4.2. [62] Let X be a nonempty subset of a normed space (X, ||.||) and *G,S : K* -> *X*. Then the pair *{G,S)* is said to be pointwise i2-weakly commuting on *K* if for a given *x e K* there exists a real number *R> 0* such that

$$\underline{\|GSx} - SGx\| < R\|Gx - \mathbf{Sx}\|$$

provided *Gx,Sx* £ *K*.

If above inequality holds for all *x E K*, then the pair of mappings is said to be il-weaWy commuting on *K*. i2-weakly commuting mappings are pointwise *R*-weakly commuting but converse need not be true (see Example 6.4.4). Notice that for K" = X and *R* = *l*, i?-weak commutativity reduces to weak commutativity for self mappings due to Sessa [136]. Here it may be noted that i2-weak commutativity implies weak compatibility at the points of coincidence and remains a minimal condition to obtain results on conunon fixed points.

The classes of Cq-commuting and J?-weakly commuting mappings are different. The following examples demonstrate the situation better.

The pair of mappings in Example 6.4.1 is i?-weakly commuting but not *Cg*-commuting.

EXAMPLE 6.4.2. Let X = K with usual norm and *K* = *[1,2]*. Define *G,S* : *K*-^*X* *as*

r a;2^ if 1< x < 2 T 22;, if 1< x < 2
Gx = < and 5a; = <
1^ 1, ifx = l [1, if xG{l,2}.

K is g-starshaped set with g = 1 G *F{S)* and *Cq{G,S)* - {1}. It is easy to verify that the pair (G, *S)* is C,-commuting but not /2-weakly commuting.

However, there do exist pair of mappings which possesses both the properties at the same time and otherwise as well.

EXAMPLE 6.4.3. Let X = 3? with usual norm and *K* = [0, |]. Define *G,S* : *K*-^*X* *as*

(1 + ^, if X e {0} u (1, |]
' 1, if O < x < l
Sx= < and Gx = ^ |, if 0 < x < 1
. *I* if l < ^ < t
^ 1 , ifa; = l.

Clearly, *K* is g-starshaped set with *q* = *l e F{S)* and *Cg{G, S)* = {0,1}. It is easy to verify that the pair (G, *S)* is Gg-commuting as well as i?-weakly commuting.

EXAMPLE 6.4.4. Let X = 3? with usual norm and *K* = [0,1]. Define *G,S* :

[0,l]^[|,f)cKas

5a; = { |, if 0 < a; < 1 and $Gx=$ < ' f + ^, if 0 < X < 1
v $\frac{3}{4}$', if X = 1.

First, note that if x G [0, -4] then *Gx, Sx* € [0,1] = *K.* One can show that (G,S) is not i?-weakly commuting on [0, -^], for one cannot find R > 0 satisfying the definition of il-weak commutativity. But, for some x G [0, -^*],* one can always find some *R>* *0* satisfying the definition of pointwise i2-weak commutativity. For instance if we take x = r^, then

SO V2v/2J **GS** .2V2, <*R* \2V2J *G* ,2V2. 1 ^

holds for all i? ≥ I 2-

Notice that *K* is 9-starshaped with 9 = | e *F{S)* and *C{G, S)* = {0,1} but the pair *{G, S)* is not weak compatible and hence not Cq-commuting on *K.*

REMARK 6.4.1. it is straightforward to note that the pair *{G, S)* of Cq-commuting mappings is /2-weakly commuting on *Cq{G,S)* and hence pointwise i?-weakly commuting. But converse need not be true (see Example 6.4.4).

Now we state and prove the following theorem in Banach spaces as an apphcation of our main theorem.

THEOREM 6.4.1. Let *K hea.* nonempty weakly compact qi-starshaped subset of a Banach space *X* and (F, *G)* be a generalized S'-nonexpansive mappings of *K* into *X* (with <*f)(t) = t)* satisfying

(/24) *dK* c *SK,* *{FK*U*GK)* *nKcSK,*

(/25) *SxedK=>* *Fx, Gx* *e* *K,*

(/ge) the pair *{G, S)* is compatible and Cq-commuting.

Moreover, if (/ — *G)* is demiclosed and *S* is affine, then the mappings F, *G* and *S* have a common fixed point in *K* provided *S* is continuous.

PROOF. Notice that (due to Lemma 6.2.1) the pair *{G,S)* is reciprocally continuous. Since *K* is 9-starshaped subset of *X,* then (1 *-t)q + tx* € *K* for all x G /<". Define a mapping *Gn : K* -¥ *X by G^x* = (1 - fcn)? + *knGx* for *aXL x e K,* where *{kn}* is a sequence in [0,1] such that fcn —^ 1- Then it is straightforward to verify that the pau: *{F,* G„) is generalized 5-contraction mappings of *K* into *X* and G„ also satisfying conditions (/24)-(/26)- Since weak topology is Hausdorff and *K* is weakly compact, therefore *K* is weakly closed and hence strongly closed. Now by Theorem 6.3.1 with (T = *S)* for each n > 2, the mappings F,G„ and *S* have a unique common fixed pomt, say *Zn.* By the weak compactness of *K,* there exists a subsequence *{zm}* of *{zn}* and *z e K* such that *Zm* -> *z* weakly. Since weakly convergent sequences are bounded, therefore, {z„^} is also bounded, i.e. there is a constant p > 0 such that ||znill < p for all n > 2. For each n > 2, we have

(7 - *G)Zn,* =*Zr*^- *K^lGr^Zn,* - (1 " *KM*

= (1 - *K')*^ + *K'* -1)9

and hence

||(/-GK||<|fc-^-l|(p+Nl).

Since fc;;^.^ ^ 1 as n ^ oo, we have (/ - *G)z* ->• 0 G X. Also *z^* -^ *z e K* and (/ - *G)* is demiclosed, it follows that (/ - *G)z* = 0 yielding thereby *Gz = z.* As 2:„. is a fixed point of *S* and *S* is continuous, then *Sz = z.* Suppose *Fz* ^ *z,* then using inequality (6.1) (for *T = S)*

d{Fz, z) = d{Fz, Gz) < a max I *]rd{Sz, Sz), d{Sz, Fz), diSz, Gz)* \

+ b[d{Sz,Gz) + d{Sz,Fz)]

d{Fz, z)<{a + b)d{Fz, Sz) < d{Fz, Sz) = d{Fz, z)

which is a contradiction. Hence z is a common fixed point of the mappings F, *G* and *S.* This completes the proof.

Chapter 7

SOME COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

7.1. INTRODUCTION

There do exist many concrete and practical situations wherein the distances between the points are rather inexact than being a single nonnegative real number which led to the introduction of probabilistic metric spaces and continues to be a subject of interest for the researchers of this domain. But if uncertainty is due to fuzziness rather than randomness, then in this situation concept of fuzzy metric spaces is relatively more suitable. Inspired from these observations, Deng [26], Erceg [29] and Kramosil and Michalek [92] introduced the notion of fuzzy metric spaces by generalizing the concept of the probabilistic metric spaces to the fuzzy situations.

On the other hand, Kaleva and Seikkala [81] generaUzed the notion of metric spaces by setting the distance between the points to be nonnegative fuzzy numbers where triangle inequality is realized by defining an ordering in the set of fuzzy numbers. This natural way of defining fuzzy metric spaces has been exploited by several researchers of this domain especially metric fixed point theorists and by now there exists considerable hterature on fixed point theorems in fuzzy metric spaces which includes [39,40,80,146]. Most recently, Xia and Guo [155] also redefined the fuzzy metric spaces using fuzzy scalers instead of fuzzy numbers or real numbers along with some resvdts on completeness of fuzzy metric spaces. The approach of Xia and Guo [155] is more natural and soothing and as per oiu: expectation, it may inspire further developments in near future.

The recent Uterature had observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probabiUty theory, logic etc. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities

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for fixed point theorists. In fact, the last three decades were very productive for fuzzy mathematics. Grabiec [43] proved first ever the fuzzy version of Banach contraction principle and Edelstein [28] fixed point theorem and by now there exists considerable hterature on this topic. To mention a few, we cite [20,22,32,52,108,152].

In 1999, Vasuki [152] proved the following common fixed point theorem for a pair of /2-weakly commuting mappings satisfying a Boyd and Wong [13] type contraction condition which is a fuzzy version of a result due to Pant [102, Theorem 1].

THEOREM 7.1.1. Let (X,M,*) be a complete fuzzy metric space, A and S be i?-weakly commuting self mappings of X satisfying $A\{X)$ C $S\{X)$ and $M\{Ax, Ay, t)$ > $r\{M\{Sx, Sy,t))$ for all x,y e X, where r : [0,1] -> [0,1] is continuous function such that $r\{s) > s$ for each 0 < 5 < 1. If one of A and S is continuous, then A and 5 have a unique common fixed point in X.

Here it may be pointed out that Theorem 7.1.1 has been further extended for two pairs of iZ-weakly commuting mappings by Chugh and Kumar [22] and Singh and Jain [140].

In this chapter, we introduce a suitable implicit function to prove fixed point theorems in fuzzy metric spaces and also furnish several examples enjoying the format of our implicit function. We are not aware of any fixed point theorem proved via implicit functions in fuzzy metric spaces. In process, several previously known results due to Chugh and Kumar [22], Imdad and Ali [52], Singh and Jain [140] and Vasuki [152] can be deduced as a special case. Moreover, adopting jR-weak commutativity of tjqje (^4/), type (Ag) to fuzzy setting and to introduce i2-weak commutativity of type (P) which are to be used to prove our results in this chapter.

The improvement realized in our results is four fold which includes:

(^i) relaxing the continuity requirement of all mappings completely,

(^2) minimizing the commutativity requirement of the mappings to the point of coincidence,

(pa) replacing the completeness of the space to four alternative natural conditions,

(^4) replacing contraction condition with a suitable implicit function to cover several conditions in one go.

The rest of the scheme of this chapter is as follows. In Section 7.2, we present some relevant definitions, results and also discuss the independence amongst certain weak conditions of commutativity. In Section 7.3, we define a new class of implicit function and furnish several examples to substantiate the worth of this cleiss. In the last section, we prove a general conunon fixed point theorem for weakly compatible mappings. Some related results and illustrative examples are also discussed.

7.2. PRELIMINARIES

In what follows, we collect relevant definitions, results and examples to make the presentation of our results as self-contained as possible.

DEFINITION 7.2.1.[156] A fuzzy set A in A'' is a function with domain X and values in [0, 1].

DEFINITION 7.2.2.[135] A binary operation • : [0,1] X [0,1] -)• [0,1] is a continuous ^norm if {[0,1],*} is an Abelian topological monoid with unit 1 such that a-kb < c-kd whenever a<c and b <d, a,b,c,d^ [0,1].

DEFINITION 7.2.3.[92] The triplet (A',M,*) is a fuzzy metric space if X is an arbitrary set, • is a continuous <-norm, and M is a fuzzy set in X^ x [0, oo) satisfying the following conditions:

(55) M(x,y,0) = 0,

(ge) *M{x, y,t)* = *1* for alU > 0 iff x = y,

(37) *M{x,y,t)* = *M{y,x,t),*

(gs) *M{x, y, t)* • M(y, *z, s)* < *M{x, z, t + s),*

(gg) *M{x,y,.) :* [0,*00)* -> [0,1] is left continuous, for *a]lx,y,z eX* and *s,t>Q.*

In recent years, George and Veeramani [39] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [92] and defined Hausdorff topology of metric spaces which is later adjudged metrizable. They also showed that every metric induces a fuzzy metric and furnished the following example (in sense of George and Veeramani [39]).

EXAMPLE 7.2.1. To every metric one can always associate a fuzzy metric. To do this, let *{X,d)* be a metric space, $a*b = ab$ and $M(x,y,t) = $ -——T/—\> fct" + *ma{x,y)*

fc, m, n, t e 3?'*'. Then (X, M, •) is a fuzzy metric space. If we put fc = m = n = 1, we get

$$M\{x,y,t) = t + d(x,y)"$$

The fuzzy metric induced by a metric *d* is also sometimes referred as standard fuzzy metric.

DEFINITION 7.2.4.[43] A sequence {x„} in a fuzzy metric space (X,M,*) converges to a; e X if

$$\lim_{n\to oo} M\{x_{,,}, x, i) = 1 \quad \text{for each } t > 0.$$

Recently, there is some debate on existing definitions of Cauchy sequences which are available in [146,153] wherein the Cauchy sequence defined by Grabiec [43] is labeled as G-Cauchy sequences. But in order to prove our results, we adopt the definition of Cauchy sequence in the sense of Vasuki and Veeramani [153].

DEFINITION 7.2.5.[153] A sequence {x„} in a fuzzy metric space (X.M,*) is called G-Cauchy if $\lim_{n\to cx3}$ *M{xn+p,* x„, f) = 1 for every < > 0 and each p > 0. Moreover, *{X, M,* •) is called G-complete if every G-Cauchy sequence in *X* converges in *X.*

DEFINITION 7.2.6. A pair of self mappings *{S,T)* of a fuzzy metric space *{X, M,* •) is said to be

(gio) weakly commuting (cf. [152]) *li M{STx,TSx,t) > M{Sx,Tx,t),*

(gn) i?-weakly commuting (cf. [152]) if there exists some *R> 0* such that *M{STx, TSx,t)>M{Sx,Tx,t/R),*

{912) /2-weakly commuting mappings of type *(Af)* if there exists some *R> 0* such that *M{STx,TTx,t)* > *M{Sx,Tx,t/R),*

(^is) i?-weakly commuting mappings of type *(Ag)* if there exists some $R > 0$ such that $M(SSx,TSx,t) > M(Sx,Tx,t/R)$,

(du) /?-weakly commuting mappings of type (F) if there exists some $R > 0$ such that $M(SSx,TTx,t) > M(Sx,Tx,t/R)$, for aU a; € X and f > 0.

Notice that Definition 7.2.6(pi2) and Definition 7.2.6(513) axe inspired by Pathak et al. [118] whereas we are not famiUar with Definition 7.2.6(514) and thus far it seems to be a new entry to the literature.

EXAMPLE 7.2.2.(cf. [152]) Let X = ^, the set of real numbers. Define a-kb ^ ab and

$$\begin{cases} (e'^)'^ & \text{forallx,j/eXandi>0} \\ 0, & \text{for all X, y G Xand } t = 0. \end{cases}$$

Then it is well known that $(X, M, \cdot)$ is a fuzzy metric space. Define self mappings on X as $Sx = 2x - 1$ and $Tx = x$^. Then by a straightforward calculation, one can show that

$$\begin{aligned} M(STx,TSx,t) &= f e"^ \quad \text{21x-lf} \setminus \\ &= M(Sx,Tx,t/2) \end{aligned}$$

which shows that the pair (S,T) is i?-weakly commuting with $R=2$. Note that the pair (S, T) is not weakly commuting due to strict increasing property of exponential function.

However, various kinds of above mentioned /2-weak commutativity notions are independent of one another and none impUes the other. The earlier example can be utilized to demonstrate this inter independence.

To demonstrate the independence of i?-weak commutativity with K-weak commutativity of type *(Af)*, notice that

$$\begin{aligned} M(STx,TTx,t) &= le—i—^j = (e ' « ' ^ «' j \\ &< (e ^ ^) = M(Sx, Tx, t/R) \text{ when } x > 1 \end{aligned}$$

which shows that /2-weak commutativity does not imply il-weak commutativity of type *(Af)*.

Secondly, in order to demonstrate the independence of K-weak commutativity with i2-weak commutativity of type (P), note that

$$M\{SSx,TTx,t)=U\wedge\text{—}r\text{-}\wedge\backslash \quad = \quad fe \quad \wedge \quad ' \quad ' \quad « \quad 'j$$

$$< \ (e\wedge\wedge\wedge \quad = \ M\{Sx,\ Tx,\ t/R) \ \text{ for } \ x > 1.$$

Finally, the pair (5, *T)* is il-weakly commuting of type *{Ag)* as

-1

which shows that the pair (5, *T)* is i?-weakly commuting of type *{Ag)* with /? = 4. This situation may also be utiUzed to interprete that an i2-weakly commuting pair of type *{Ag)* need not be i?-weakly commuting pair of type *{Af)* or type (F). It is not difficult to find examples to establish the independence of one of these definitions from the others which shows that there exist situations to suit a definition but not the others.

However, the /2-weak commutativity of type *{Af)*, type *{Ag)* and type (P) can together imply P-weak commutativity in a specific setting which can be described as follows:

PROPOSITION 7.2.1. Let *{S,T)* be a pah: of self mappings which is i?-weakly commuting of type *{Af)*, type *{Ag)* and type *{P)* (at the same time) and $a*b$ = min{a,*b}*, then the pair *{S,T)* is P-weakly commuting.

PROOF, it is straightforward to write

$$M\{STx,\ TSx,\ t) \ > \ M\{STx,\ TTx,\ t/3) \bullet M\{TTx,\ SSx,\ t/S) \bullet M\{SSx,\ TSx,\ t/3).$$

Now using the definitions of i?-weakly commuting of type *{Af)*, type *{Ag)* and type (P) there exists constants *Ri,R2,R3>0* satisfying

$$M\{STx,\ TSx,\ t) \ > \ M\{Sx,\ Tx,\ t/ZRi) \bullet M\{Sx,\ Tx,\ t/ZRi) \ * \ M\{Sx,Tx,\ t/SRs)$$

implying thereby

$$M(STx,TSx,t) \quad > \quad M(Sx,Tx,t/3Ri)$$

(for some 1 < i < 3) which shows that the pair *{S,T)* is i?-weakly commuting.

7.3. IMPLICIT FUNCTION

As mentioned time and again in this text, Popa [122,124] and Imdad et al. [63] used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its unifying power as an implicit function can cover several contraction conditions at the same time which includes known as well as imknown contraction conditions. In this section, we also define a suitable implicit function in fuzzy metric spaces to prove our results. Let Q denote the family of all continuous functions *F* : [0,1]^ -f 3? satisfying the following conditions:

F\ : For every *u* € (0,1] and u e [0,1] with *F{u,v,u,v)* > 0 or *F{u,v,v,u)* > 0, we have *u* > *v.*

F2: *F{u, u,* 1,1) < 0, for all *u* e (0,1].

EXAMPLE 7.3.1. Define F(ti,...,t4): [O,!]"-> Sfl as

$$F\{tut2,t3,t4) = ti - (t)\{mm\{t2,t3,U\}),$$

where *4>* : [0,1] ->• [0,1] is a continuous function such that *<f){s)* > s for 0 < s < 1. Then

Fi : *F{u, v,u,v)* = *u* — ^(min{u, *u, v})* > 0.
If « < u, then *u* — 0(w) > 0 imply *u* > *(j){u)* > u, a contradiction. Hence *u*> *v.*

Fa : *F{u,* u, 1,1) = w - *4>{mm{u,* 1,1})
= u - *<f){u)* < 0, for all u > 0.

EXAMPLE 7.3.2. DefineF(ii,...,f4) : [0,l]''^3fias

•^(^1,^2, ^3,1^4) = *ti*- A;min{i2,*3,^4}, where A; > 1.

Fi : *F{u,v,u,v) = u — kvam{v,u,v} >Q.*

liu<v, then w > fcu > w, a contradiction. Hence *u> v.*

F2 : F{u,u,l,l) =u — kxam{u,l,l}

= *u{l -k)<0,* for all u > 0.

EXAMPLE 7.3.3. Define *F{ti,...,^4):* [0,1]" -»• 3? as

F{ti,h,tz,U) = ti- kti - min{t3,t4}, where fc > 0.

F\ : F{u,v,u,v) = u- kv- min{w,*v}* > 0.

li u <v, then *kv <0, a.* contradiction. Hence *u> v.*

F2 : F{u,u,l,l) =u-ku- min{l, 1}

= (1 - A;)u - 1 < 0, for all u > 0.

EXAMPLE 7.3.4. Define *F{ti,...,U):* [O,l]" -^ 3? as

F(tut2,h,U) = *i - *at2* - *btz* - *ct^,*

where a > 1 and 6,c > 0(7^ 1).

EXAMPLE 7.3.5. DefineF(ii,...,f4): [o,i]"^3f?as

F(ti, t2,^3,^4) = ti - *at2* - *h{tz* + ^4),

where a > 1 and *b* > *0{*^ 1).

EXAMPLE 7.3.6. Define F(«i,... ,^4): [0,1]^ ^ 3e as

F(ti,t2,ti,t4) = tf - *kt2tzU,* where A; > 1.

Since verification of requirements (Fi and *F2)* for Examples 7.3.4 -7.3.6 is straightforward, hence details are omitted.

7.4. MAIN RESULTS

Now we state and prove our results as follows:

THEOREM 7.4.1. Let *A,B,S* and *T* be four self mappings of a fuzzy metric space *{X, M,* •) satisfying the condition

F{M{Ax,By,t),M{Sx,Ty,t),M{Sx,Ax,t),MiBy,Ty,t)) > 0 (7.1) for all distinct x, y € X smd t > 0, where F 6 ft. Suppose that

(^15) *AiX)* C *T{X)* and *B{X)* C *S{X),*

(gie) one of *A{X), B{X), S{X)* or *T(X)* is a complete subspace of *X.*

Then the pairs *{A,S)* and (B,T) have points of coincidence. Moreover, if the pairs *{A, S)* and (B, T) are weakly compatible, then *A, B, S* and *T* have a unique common fixed point.

PROOF. Let xo be an arbitrary point in *X.* Then following arguments of Fisher [37], one can construct sequences {x„} and {y„} in *X* such that

$$y2n = Tx2n+l = Ax\text{-}jn \text{ and } y2n+l = SX2n+2 = \text{-}BX2n+l\text{-}$$

The sequences {xn} and {y„} in *X* are such that x„ ->^ x,y„ -)• y,^ > 0 implies *M(xn,yn,t)* -^ *M{x,y,t).*

Now making use of (7.1), we have

$$F\{M\{Ax2n, \quad BX2n+ut), \quad M\{SX2n, \quad TX2n+ut), \quad M(5X2„, \quad AX2n, \quad t),$$

$$M\{Bx2n\text{^}l,TX2n+ut))>0$$

$$\text{or} \qquad F\{M\{y2n,y2n+ut),M\{y2n\text{-}Uy2n,t),M\{y2n\text{-}uy2n,t),M\{y2n,y2n+ut)) \qquad > \qquad 0\text{-}$$

Hence in view of (Fi), we have

$$M(y2„,y2„+i,t) > M(y2„\text{-}i,y2n,t)\text{-} \tag{7-2}$$

Thus *{M(y2n,y2n+i,t),n* > 0} is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a Umit *I* < *I.* We assert that *1* = *1.* If not, (i.e. *I* < 1) then on letting n -^ oo in (7.2) one gets *I* > *I, o.* contradiction. Hence / = 1. Therefore for every *n E N,* using analogous arguments one can also show that *{M(y2n+i,y2n+2, t),* n > 0} is a sequence of positive real numbers in [0,1] which converges to 1. Therefore for every *n e N*

$$M\{yn,yn+i,t) > M\{yn\text{-}i,yn,t) \text{ and } \lim_{n\to\infty} M(y„,y„+i,t) = 1.$$

Now for any positive integer *p*

$$Miyny\, Vn+p, t) > M\{yn, yn+i,t/p) \bullet \ldots \bullet M(y„+p\_i, y„+p, t/p).$$

Since $\underset{n—¥oo}{\text{Una}}$ $M\{yn,yn+ut)$ = 1 foi t > 0, it follows that

$$\lim_{n\to oo} M(y_„,y_„+p,f) > 1*1*\ldots*1 = 1$$

which shows that {y„} is a Cauchy sequence in *X*.

Now suppose that *S{X)* is a complete subspace of *X,* then the subsequence *{y2n+\}* must converge in *S{X).* Call this limit to be u and *v* € *S~^u.* Then S'u = tt. As {t/„} is a Cauchy sequence containing a convergent subsequence {r/2n+i}, therefore the sequence {y„} also converges implying thereby the convergence of {yzn} being a subsequence of the convergent sequence {j/„}. If *Av* ^ *Sv,* then on setting *x* = *v* and *y* = *X2n+i* in (7.1), one gets (for *t* > *0)*

$$FiM(Av,Bx2n+utlM\{Sv,\text{Txan+i,«}),M\{Sv,Av,«),M(\text{Bx2„+i,Tx2n+i},t)) \quad > \quad 0$$

which on letting *n* —> *oo* reduces to

$$F\{M\{Av, u, t), M\{Sv, u, t), M\{Sv, Av,t), M\{u,u, t)) > 0$$

$$\text{or} \qquad F\{M\{Av, Sv, t), 1, M(Su, Av, t), 1) > 0$$

yielding thereby, *M{Av,Sv,t)* > 1, a contradiction. Hence, *Av* = *Sv* which shows that the pair *{A, S)* has a point of coincidence.

As *A{X)* C *T{X)* and Au = u implies that *u* £ *T(X).* Let *w* € T'^u, then Tu; = *u.* Suppose that *Tw* ^ *Bw.* Again using (7.1), we have

$$FiM\{Ax2n, Bw, t), MiSx2n, Tw, t), MiSx2n, Ax2n, t), M\{Bw, Tw, t)) > 0$$

which on letting n —> oo reduces to

$$F\{M\{Tw, Bw, t), 1,1, MiTw, Bw, t)) > 0$$

implying thereby, *M{Tw,Bw,t)* > 1, a contradiction. Hence *Tw* = *Bw.* Thus we have *u* = *Av* = *Sv* = *Bw* = *Tw* which amounts to say that both the pairs have a point of coincidence. If one assumes *T(X)* to be complete, then analogous arguments establish this claim.

The remaining two cases pertain essentially to the previous cases. Indeed, if *A{X)* is complete then « *e A{X)* C *T{X)* and if *B{X)* is complete then *u* 6 *B{X)* c *S{X).* Thus again the conclusions are completely established.

Moreover, if the pairs *{A,S)* and *{B,T)* are weakly compatible at *v* and *w* respectively, then

$$Au = ASv = SAv = Su \quad \text{and} \quad Bu = BTw = TBw = Tu.$$

If *Au* 7^ It, then for i > 0

F{M{Au, Bw, t), M{Su, Tw, t), M(Su, Au, t), M(Bw, *Tw, t))* > 0

or *F{M{Au, u, t),* M(An, u, *t),* 1,1) > 0

which contradicts *(F2).* Hence *Au = u.* This shows that u is a common fixed point of the pair *{A,S).*

Suppose that *Bu* ^ *u.* Then \ising inequality (7.1), we have

F{M(Av,Bu,t),M{Sv,Tu,t),M{Sv,Av,t),M{Bu,Tu,t]) > 0

or *F{M{u,Bu,t),M{u,Bu,t),1,1) >0*

which contradicts (F2). Hence *Bu = u.* This shows that tt is a common fixed point of the pair *{B, T).* Thus u is a common fixed point of *A, B, S* and *T.* The uniqueness of common fixed point follows easily. Also *u* remains the unique common fixed point of both the pairs separately. This completes the proof.

By setting *B = A* and *T = S in* Theorem 7.4.1, we have the following corollary for two mappings.

COROLLARY 7.4.1. Let *A* and *S* be two self mappings of a fuzzy metric space *(X, M,* •) satisfying the condition

F{M(Ax, Ay, t), M{Sx, Sy, t), M{Sx, Ax, t), M{Ay, Sy,t)) > 0 (7.3)

for all *x,y e X* and *t > 0,* where *F efl.* Suppose that

(gn) A(X) c *S{X),*

(gis) one of *A(X)* or *S(X)* is a complete subspace of *X.*

Then the pair *(A, S)* has a point of coincidence. Moreover, if the pair *(A, S)* is weakly compatible, then *A* and *S* have a unique common fixed point.

Similarly, one can obtain two corollaries for two pairs comprised of three mappings. Here we state the one obtained by setting T = 5 in Theorem 7.4.1.

COROLLARY 7.4.2. Let *A,B andS* be self mappings of a fuzzy metric space *{X, M,* •) satisfying the condition (pis) and

F{M(Ax, By, t), M(Sx, Sy, t), M(Sx, Ax, t), M{By, Sy, t)) > 0 (7.4)

for all *x,y e X* and *t> 0,* where F € fi. Suppose that

{gig)A{X)L)B{X)cS{X).

Then the pair *(A, S)* as well as *{B, S)* has a point of coincidence. Moreover, if the pairs *{A, S)* and (B, *S)* are weakly compatible, then *A, B* and *S* have a unique common fixed point.

COROLLARY 7.4.3. The conclusions of Theorem 7.4.1 remain true if for all distinct *x,y e X* inequality (7.1) is replaced by one of the following:

(P20) *M{Ax,By,t)* > *(t>{ixdn{M{Sx,Ty,t),M{Sx,Ax,t),M{By,Ty,t)}),*

where *(p* : [0,1] —> [0,1] is a continuous function such that *<f){s)* > s for all 0 < s < 1.

(P2i) *M(Ax,By,t)* > *kmm{M{Sx,Ty,t),M{Sx,Ax,t),M{By,Ty,t)},*

where A; > 1.

{922) *M{Ax, By, t)* > *kM{Sx,Ty, t)* + min{M(5'x, *Ax, t), M{By, Ty,t)},*

where fc > 0.

(^23) *M{Ax, By, t)* > *aM(Sx, Ty, t)* + *bM{Sx, Ax, t)* + *cM{By, Ty, t),*

where a > 1 and *b,c> 0{*^ 1).

{92i) *M{Ax, By,t)* > *aM{Sx,Ty, t)* + *b[M{Sx, Ax, t)* + *M{By,Ty, t)],*

where a > 1 and $b > 0\{$^ 1).

(^25) $M^\{Ax,By,t) > kM\{Sx,Ty,t)M\{Sx,Ax,t)\{By,Ty,t),$ where fc > 1.

PROOF. The proof of the corollaries corresponding to contraction conditions 520-^25 follows from Theorem 7.4.1 and Examples 7.3.1-7.3.6.

REMARK 7.4.1. Corollary corresponding to contraction condition (^20) is a result due to Imdad and AU [52] and also generalized form of certain results contained in [22,140,152]. We also point out that some of above corollaries are new to the Uterature (e.g. Corollaries corresponding to the conditions ^21 — 525)-

THEOREM 7.4.2. Theorem 7.4.1 remains true if weak compatibility property is replaced by any one of the following (retaining the rest of the hypotheses):

(526) i?-weakly commuting property,

{927) i?-weakly conunuting property of type *(Af),*

(to) i?-weakly commuting property of type *{Ag),*

(529) i?-weakly conunuting property of type (P),

(P3o) weakly commuting property.

PROOF, since all the conditions of Theorem 7.4.1 are satisfied, therefore the existence of coincidence points for both the pairs is guaranteed. Let x be an arbitrary point of coincidence for the pair (A, S), then using i?-weak commutativity one gets

$$M\{ASx, SAx, t) > M\{Ax, Sx, t/R) = 1$$

which amounts to say that $ASx = SAx$. Thus the pair $\{A, S)$ is weakly compatible. Similarly $\{B,T)$ commutes at all of its coincidence points. Now appealing Theorem 7.4.1, one concludes that A, B, S and T have a unique common fixed point.

In case (A, S) is i?-weakly commuting pair of type (Af), then

$$M\{ASx,S^x,t) > M\{Ax,Sx,t/R) = 1$$

which amounts to say that $ASx = S^\wedge x$. Now

$$M\{ASx, SAx, t) > M(ASx, S^\wedge x, t/2) * M\{S^\wedge x, SAx, t/2)$$

$$= 1*1 = 1$$

yielding thereby $ASx = SAx$. Similarly if the pair $\{A, S)$ is /?-weakly commuting mapping of type (Ag) or type (P) or weakly commuting, then the pair $\{A, S)$ also commutes at the points of coincidence. Similarly one can show that the pair $\{B,T)$ also commutes at the points of coincidence. Now in view of Theorem 7.4.1, in all four cases A, B, S and T have a unique common fixed point. This completes the proof.

As an appUcation of Theorem 7.4.1, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

THEOREM 7.4.3. Let $\{Ai,A2,...,Am\},\{Bi,B2,...,Bn\},\{Si,S2,...,Sj,\}$ and $\{ri,r2,\ldots,T,\}$ be four finite families of self mappings of a fuzzy metric space (X,M,*) such that $A = AiA2...Am$, $B = BiBi\text{-.-}Bn$, $S = SiS2...Sp$ and $T = T1T2...Tq$ satisfy inequality (7.1), conditions (pis) and (pie)-

Then the pairs (A, S) and $\{B, T)$ have points of coincidence.

Moreover, if Ai>lj=AjAi, $BkBi = BiBk$, 5r5, = 5 A , $TtTr, = T^\wedge Tt$, $AiSr = SrAi$and$BkTt = TtBk$foralH, j € A = {1,2,...,m}, fc,/ G /a = {1,2,...,n}, r,se $h = \{1,2,\ldots ,p\}$ and $t,uE\ h = \{1,2,..., q)$, then (for $dXM\ e\ h,k\ e\ h^\wedge r \wedge h$ and $t\ El/^\wedge\ Ai,Sr,Bk$ and Tj have a common fixed point.

PROOF. The proof follows on the Unes of a result due to Imdad and Ah [, Theorem 3.3], hence details are avoided.

By setting $A = A^{\wedge\wedge} A2 = ... = Am,B = Bx = B2 = ... = Bn,S \wedge Si = S2$-- ...$= Sp$ and $T = Ti=T2 = ... = TqVD$. Theorem 7.4.3, one deduces the following for certain iterates of mappings.

COROLLARY 7.4.4. Let A, B,S and r be four self mappings of a fuzzy metric space (X, $M,\text{-}k)$ such that A"*, S",5^ and $T^\wedge$ satisfy the condition (7.1). If one of

A'^iX), B'^iX), S^iX) or *T^{X)* is a complete subspace of *X,* then *A, B, S* and *T* have a unique common fixed point provided (.A, *S)* and *{B, T)* commute.

REMARK 7.4.2. Results similar to Corollary 7.4.3 may be obtained in respect of Theorems 7.4.2, 7.4.3 and Corollary 7.4.4. But due to the repetition, we did not included the details.

The following example furnishes an instance where Corollary 7.4.4(^2o) is appUcable but Theorem 7.1.1 (also theorem due to Chugh and Kiunar [22]) cannot be used due to the absence of continuity requirement.

EXAMPLE 7.4.1. Consider *X* = [0,1] equipped with the natural metric *d{x, y)* = *\x-y*. Now for *t G* [0, oo) define

' 0, if t = Oandx,y GX

M{x,y,t)=<

j ^ , ifi>Oandx,yGX.

Clearly (X, M, *) is a fuzzy metric on *X* where * is defined as Q * 6 = ab.

Define *A, B,S andT* on [0, 1] as

1, ifxG[o,i]nQ C 1, ifxe[0,i]n(5

Ax= (Bx = <

0, ifx^[o,i]nQ [*I* ifx^[o,i]nQ

i, ifO<x<l M, ifO<x<l

Sx= (and Tx = <

1, ifx = l (^ 1, ifx = l.

Then A2(X) = {1} C {J, 1} = *T\X)* and *B^{X)* = {1} C {i, 1} = *S^{X).* Define *(l>*: [0,1] -^ [0,1] as 0(0) = 0, <^(1) = 1 and *(p(s)* = *y/s* for all *s* € (0,1). Then

l = *M{A^x,B\t)*

> *<l>{mm{M{S^x, T\ t), M{S\ A\ t), M{B\ T'^y, t)})*

for all f > 0. Also the various componentwise commutativity conditions ensure the commutativity of the both pairs *{A,S)* and *{B,T).* Thus all the conditions of the Corollary 7.4.4 are satisfied and 1 is the common fixed point of *A, B, S* and *T.*

Here one needs to note that Theorem 7.1.1 (also theorem due to Chugh and Kumar [22]) cannot be used in the context of this example because if we take *x,y* ^

Q, then

$$t$$

which is not always true for $t > 0$ (e.g. $t = 0.5$). On the other hand all the four mappings are discontinuous which is not in lieu of the requirements of the Theorem 7.1.1 (also theorem due to Chugh and Kumar [22]).

Finally, we furnish an example to create a situation which demonstrates the utihty of Theorem 7.4.3.

EXAMPLE 7.4.2. Consider (A",M,*) as m Example 7.4.1. Define four finite families of self mappings as

$$A_n x = \begin{cases} 1, & \text{if } x \in [0,1] \cap Q \\ i, & \text{if } x \notin [0,1] \cap Q, \end{cases} \qquad B_n x = \begin{cases} 1, & \text{if } x \in [0,1] \cap Q \\ i, & \text{if } x \notin [0,1] \cap Q, \end{cases}$$

$$S_n x = \begin{cases} 1, & \text{if } x \in [0,1] \cap Q \\ \wedge, & \text{if } x \notin [0,1] \cap Q \end{cases} \quad \text{and} \quad T_n x = \begin{cases} 1, & \text{if } x \in [0,1] \cap Q \\ I, & \text{if } x \notin [0,1] \cap Q, \end{cases}$$

where $n = 1,2,\ldots, 100$.
Evidently, $A(X) = A_1A_2...A_{100}(X) = T_1T_2...T_{100}(X) = T(X)$ and $B(X) = B_1B_2...B_{100}(X) = S_1S_2...S_{100}(X) = S(X)$. Define $\phi : [0,1] \to [0,1]$ as in Example 7.4.1.

Retaining the same implicit function as adopted in Example 7.4.1, by routine calculations one can easily verify that the condition (7.1) is satisfied for all distinct $x,y \in [0,1]$. Also the various componentwise commutativity together ensures the commutativity of both the pairs (A,S) and (B,T). Thus all the conditions of Theorem 7.4.3 are satisfied and 1 is the common fixed point of A, B, S and T. Notice that all the component mappings of the four involved families are discontinuous.

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